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ON THE CONSTRUCTION OF RIEMANN MATRICES I[1]

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INTRODUCTION

1. Period matrices and linear associative algebras. Let

$$\omega = \| \omega_{j\nu} \| \ (j = 1, \cdots, p; \nu = 1, \cdots, 2p),$$

be the period matrix of any $2p$-ply periodic meromorphic function of p independent variables. Then (Scorza) ω is a *Riemann matrix of genus* p and possesses multiplications

$$a: \qquad \alpha\omega = \omega A,$$

where α and A are square matrices, A has rational elements, and α has complex elements. The set D of all the multiplications of ω is a *linear associative algebra over the field* R of all rational numbers and is called the *multiplication algebra of* ω. In particular, if all the matrices A are non-singular then D is a division algebra and ω is pure.

The problems of the classification and construction of Riemann matrices are of great importance in several related branches of mathematics.[2] Both problems have been reduced to the case of pure Riemann matrices but the methods of treatment used up to the present have not been quite as effective as may be desired. For, although many important and general results have been obtained, even the case of pure Riemann matrices of genus four has not yet been completed.

2. Riemann matrices and related topics. The theory of the multiplications of Riemann matrices forms the principal part of several topics in algebraic geometry.

Consider, for example, the Riemann surface M of genus p defined by an algebraic function to which belongs an algebraic curve C^p. On M there are p linearly independent integrals of the first kind, each with $2p$ periods $\omega_{j\nu}$. The p-rowed and $2p$-columned matrix

$$(1) \qquad \omega = \| \omega_{j\nu} \|$$

[1] On Feb. 28, 1931 I presented a classification of all pure Riemann matrices of genus $p \leqq 8$ to the American Mathematical Society. The present paper, as well as a second part to follow soon, form a revision and amplification of the 1931 unpublished results.

[2] For detailed references, notably to the earlier literature and in particular to the work of Scorza, Rosati and Lefschetz, see the latter's comprehensive report, Chapters XV–XVII of the Report of the Committee on Rational Transformations, Bulletin of the National Research Council, 63 (1928), pp. 310–392.

is a Riemann matrix of genus p. The theory of the multiplications of ω is then a study of the so-called singular correspondences on C^p.

A more general question is studied when one considers the periods of theta functions or of $2p$-ply periodic functions. Here again the period matrices form Riemann matrices. But now conversely, given any Riemann matrix ω, one can define a set of $2p$-ply periodic meromorphic functions of p independent variables with ω as period matrix. In terms of this set of functions one can define an abelian variety V_p. The theory of the multiplications of the attached ω is in this case the theory of algebraic correspondences on V_p. It is in fact the systematic treatment of this problem that led to the study of Riemann matrices.

3. **Reduction to the case of pure Riemann matrices.** The problem of finding the multiplications of any Riemann matrix is therefore of great importance in algebraic geometry as well as of intrinsic interest. The first step in the solution of the problem is to reduce it to the case of pure Riemann matrices. To introduce properly this reduction we shall be compelled at this point to give the following fundamental definitions.

A matrix ω with p rows and $2p$ columns of complex elements is called a Riemann matrix of genus p if there exists a rational skew-symmetric matrix C of $2p$ rows such that

$$\omega C\omega' = 0 \text{ and } \sqrt{-1}\omega C\bar{\omega}'$$

is a positive definite Hermitian matrix. Two Riemann matrices ω_1 and ω_2 of the same genus p are called isomorphic if there exists a non-singular rational matrix G and a non-singular complex matrix γ such that $\omega_2 = \gamma\omega_1 G$. A $2p$-rowed square matrix A with rational elements is said to define a multiplication a of ω if there exists a complex p-rowed matrix α such that

$$a: \qquad \alpha\omega = \omega A.$$

The matrices A are called projectivities of ω, the matrices α substitutions of ω, and the equations a multiplications of ω. The set of all projectivities of a Riemann matrix is closed under addition, multiplication, and multiplication by rational numbers, and hence forms a linear associative algebra (of matrices) over the field R of all rational numbers. In particular the identity quantity of this algebra is the $2p$-rowed identity matrix, the zero quantity the $2p$-rowed zero matrix. The set D of all multiplications of ω is an abstract algebra with a modulus (identity element) over R and the algebra of projectivities is equivalent to D and forms a representation of D by $2p$-rowed rational matrices. The set of all substitutions of ω is also an algebra equivalent to D and forms a representation of D by p-rowed complex matrices.

A Riemann matrix ω is called impure if it has a projectivity A with zero determinant, otherwise pure. The following reduction of any Riemann matrix

(impure) to a canonical form involving only pure Riemann matrix components has been given.[3] Any Riemann matrix ω is isomorphic to a matrix

$$\left\| \begin{matrix} \Omega_1 & & 0 \\ & \ddots & \\ 0 & & \Omega_t \end{matrix} \right\|$$

in which each Ω_j is pure or

$$\Omega_j = \left\| \begin{matrix} \omega_j & & \\ & \ddots & \\ & & \omega_j \end{matrix} \right\|,$$

where ω_j is a pure Riemann matrix not isomorphic to ω_k for $j \neq k$. Moreover the multiplications of ω are uniquely determined by and expressible trivially in terms of the multiplications of the pure ω_j. For in general the algebra D is a semi-simple algebra which is a direct sum of simple algebras D_j, these being the multiplication algebras of the Ω_j. Each D_j is the *direct product* of a *total matric algebra* whose degree (corresponding to the number of rows in an equivalent algebra of matrices) is the number of ω_j in the above canonical form, *and a division algebra* which is *the multiplication algebra of* ω_j. Hence the problems of construction and classification of Riemann matrices have been reduced to the case where ω is pure.

4. **Pure Riemann matrices.** Let ω be a pure Riemann matrix so that every projectivity of ω has an inverse. It is easily seen that this inverse is also a projectivity of ω. Then the algebra D of all the multiplications of ω (equivalent to the algebra of all its projectivities) is a division algebra, that is, every quantity of D, an algebra with a modulus, has an inverse in D. We call D the multiplication algebra of ω and its order h the multiplication index of ω. The set of all quantities of D commutative with every quantity of D is a commutative division algebra or algebraic field of order ϵ over R and is called the centrum of D. As is well known, every division algebra D when represented as an algebra over its centrum has order n^2. Hence $h = n^2\epsilon$ and D is a normal division algebra of order n^2 over its centrum $R(x)$, the field of all rational functions with rational coefficients of a multiplication x of ω commutative with every multiplication of ω.

In 1929 C. Rosati[4] obtained some important results on the structure of D.

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[3] See the report by Lefschetz for several references to this reduction.

[4] In his Palermo Rendiconti paper, LIII, pp. 79–134.

He showed the presence of an operation S in D such that for every a and b of D and λ of R

$$(a + b)^s = a^s + b^s, (ab)^s = b^s \cdot a^s, (\lambda \cdot a)^s = \lambda \cdot a^s.$$

Rosati called a quantity a of D symmetric if $a^s = a$, and skew-symmetric if $a^s = -a$. According as the quantity x of the centrum $R(x)$ of D is or is not symmetric we call ω a Riemann matrix of the first or second kind, D an algebra of the first or second kind. For algebras of the second kind x is not arbitrary but of necessity $R(x) = R(s, q)$, where $\epsilon = 2t$, s is symmetric, the field $R(s)$ has order t, and q is skew-symmetric such that q^2 is in $R(s)$. For algebras of the first kind Rosati showed that n is unity or else is even. Here we write $\epsilon = t$, $x = s$, $R(x) = R(s)$.

Rosati also proved that if s is any symmetric quantity of D then the roots of its minimum equation are all real while if q is skew-symmetric then these roots are pure imaginary. In my subsequent Palermo Rendiconti paper[5] I obtained very simple proofs of many of Rosati's results and showed conversely that if the above conditions on the roots of the minimum equations of s or q are satisfied then the principal matrix C can be so selected that s is symmetric, q skew-symmetric. These two results determined completely the structure of the algebra D if $n = 1$, that is, if $D = F(x)$.

It is interesting to note that the only possible types of commutative multiplication algebras D are then precisely the two types considered in great detail by S. Lefschetz.[6] Thus the Lefschetz case of real multipliers is the case of fields of the first kind, while that of imaginary multipliers corresponds to fields of the second kind. Moreover he gave a canonical form for any Riemann matrix showing the presence, among all possible multiplications of ω, of the multiplications in the field $R(x)$. This obviously provided a canonical form for ω showing the presence of the centrum of D among its multiplications and thus a *partial* canonical form for a Riemann matrix with *any* multiplication algebra.

I recently extended[7] the Lefschetz canonical form and obtained the structure of pure Riemann matrices with any given division algebra B of multiplications. I also gave necessary and sufficient conditions that matrices in the canonical form shall actually be Riemann matrices with the given algebra B as its D. But existence proofs are very necessary to show that the rather complicated conditions so obtained can be satisfied. In fact it is quite likely that they cannot, in general, be satisfied for, while B is quite arbitrary, I have since proved that for algebras D of the first kind $n = 1, 2$.

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[5] LV (1931), pp. 57–115.

[6] In his Bocher prize paper in the Transactions of the American Mathematical Society, **22** (1921), pp. 327–482.

[7] See my Palermo Rendiconti paper just quoted as well as my paper in these Annals, **32** (1931), pp. 131–148.

Scorza has proved that the order h of D is at most $2p$ if ω is a pure Riemann matrix of genus p. I have recently proved that in this case h is in fact a divisor of $2p$, a far better restriction.[8] This, combined with my proof[9] that for Riemann matrices of the first kind $n = 1, 2$, has actually provided all the information necessary to construct all possible types of Riemann matrices of the first kind, and all of the second kind in the special cases $n = 1, 2$. In the present paper I shall derive very simple algebraic existence proofs of certain simple types of Riemann matrices. The matrices so obtained are matrices with algebraic number elements. In a second paper, On The Construction of Riemann Matrices II, I shall apply the results obtained here to give a classification and construction of the above types of Riemann matrices as well as a resulting classification of all pure Riemann matrices of genus $p \leq 8$.

I. PRELIMINARY THEORY OF RIEMANN MATRICES

1. **Known results.** In order to obtain theorems on pure Riemann matrices it is necessary to consider *Riemann matrices over a real field F*. We shall use the letter F throughout this paper to designate a field $F = R(\theta)$ of all rational functions with rational coefficients of a real algebraic number θ. The Riemann matrices which we shall define occur as sub-matrices of ordinary Riemann matrices and have analogous properties. Our treatment will be purely algebraic and all the properties we shall obtain will be derived for the more general case instead of the ordinary case $F = R(1) = R$.

A matrix ω with p rows and $2p$ columns of complex elements is called a Riemann matrix of genus p over F if there exists a $2p$-rowed square matrix C with elements in F such that

$$C' = -C, \quad \omega C \omega' = 0, \quad H = \sqrt{-1}\omega C \bar{\omega}'$$

is a positive definite Hermitian matrix.[10] The matrix C is called a principal F-matrix of ω. It is easily shown[11] that both C and the matrix

$$\left\| \begin{matrix} \omega \\ \bar{\omega} \end{matrix} \right\|$$

are non-singular.

A $2p$-rowed square matrix A with elements in F is said to define a multiplication a of ω if there exists a p-rowed square complex matrix α such that

a: $$\alpha\omega = \omega A.$$

[8] In my paper in the Transactions of the American Mathematical Society **33** (1931), pp. 219–234.

[9] Proved in these Annals **33** (1932), pp. 311–318.

[10] A square matrix H is said to be positive definite Hermitian if there exists a complex square matrix P such that $H = P\bar{P}'$. A necessary and sufficient condition that H be positive definite Hermitian is that $H = \bar{H}'$ and that every principal minor of H shall have positive determinant. We use here the notation H' for the transpose of H, $\bar{H}$ for the matrix whose elements are the complex conjugates of those of H.

[11] Cf. S. Lefschetz, loc. cit., p. 381 for a proof.

The set D of all multiplications of ω forms a linear associative algebra over the field F and is a division algebra when ω is pure. The set D_1 of all the *projectivities* A of ω is an algebra of $2p$-rowed square matrices with elements in F and is equivalent to D. The set D_2 of all the substitutions α of ω is an algebra of p-rowed complex square matrices over F and is also equivalent to D.

Let ω be a pure Riemann matrix so that D is a division algebra. As stated in our introduction for the case $F = R$ we now have D expressible as a normal division algebra of order n^2 over its centrum $F(x)$. The order of $F(x)$ with respect to D is ϵ and the order of D with respect to F is $h = n^2\epsilon$. The statement that D is a normal division algebra over $F(x)$ means that the only quantities of D commutative with every quantity of D are the quantities of $F(x)$. These latter quantities are all commutative with every quantity in D and are all polynomials in x with coefficients in F.

A $2p$-rowed square matrix B with elements in F is called an F-matrix of ω if $\omega B\omega' = 0$. The set of all F-matrices of ω forms a linear set Σ over F. The order of Σ (i.e., the number of matrices in Σ linearly independent with respect to F) is exactly h and in fact it is easily seen that, symbollically,

$$\Sigma = D_1 \cdot C,$$

where C is any principal F-matrix of ω.

Let A be a projectivity of ω. Rosati was first to notice that it follows immediately that, since $B = AC$ is an F-matrix of ω, so also is B', and hence $B'C^{-1}$ is a projectivity of ω. But then

$$A^* = CA'C^{-1}$$

is a projectivity of ω. This induces an operation in D such that

$$(a + b)^* = a^* + b^*, \ (ab)^* = b^*a^*, \ (\lambda a)^* = \lambda a^*,$$

for every a and b of D and λ of F. If a is in D and such that $a^* = a$ then a is called a symmetric quantity of D, $A^* = A$ a symmetric projectivity of ω. If $a^* = -a$ then a and A are said to be skew-symmetric. Rosati proved[12] that for a pure Riemann matrix ω *the roots of the minimum equation of a symmetric quantity are all real,* while *the roots of the minimum equation of a skew-symmetric quantity are all pure imaginary.*

An algebra D of order $h = n^2r$ over F is said to be an algebra of the first kind if $x^* = x$. In this case ω is a Riemann matrix of the first kind and we set $\epsilon = t$, $x = s$. In the contrary case D and ω are of the second kind, $\epsilon = 2t$, $F(x) = F(u, s)$, where $s = s^*$, $u^* = -u$, $u^2 = \sigma(s)$ in $F(s)$. In all cases $2p$ is divisible by h and nt divides p.

The singularity index k of a Riemann matrix ω over F is the maximum number of linearly independent symmetric quantities of D. This is in fact

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[12] Cf. his paper quoted in the Introduction. He considered ordinary Riemann matrices but *all real* projectivities and his proof is valid for any ω over a real algebraic field F.

the order of the sub-set of Σ which consists of all the alternate matrices in Σ. It is well to mention the fact that h and k obviously have the least value unity since at least $\omega C\omega' = 0$. Scorza put $1 + h$, $1 + k$ where we have put h and k. This is of no value as it merely complicates formulas and is completely unnecessary.

By an extremely long argument Rosati proved (loc. cit.) that for Riemann matrices of the first kind either

$$h = n^2t, \quad k = \frac{n(n+1)t}{2},$$

or

$$h = n^2t, \quad k = \frac{n(n-1)t}{2},$$

according as D does or does not contain a quantity $a = a^*$ of grade n with respect to $F(s)$. For the case of Riemann matrices of the second kind it was easily shown that $h = 2t$ and D contains a quantity $a^* = a$ of grade n with respect to both $F(x)$ and $F(s)$.[13]

I have recently proved that the multiplication algebra of every pure Riemann matrix of the first kind is either the field $F(s)$ or a generalized quaternion algebra over $F(s)$. Hence $n = 1$ or 2 and, as I shall show in my second paper, the determination of the values of k becomes trivial. The values themselves are now $h = k$ when $n = 1$, $h = 4t$, $k = 3t$ for an algebra over $F(s)$ containing a symmetric quantity not in $F(s)$, and finally $h = 4t = 4k$ for a generalized quaternion algebra over $F(s)$ which contains no quantity symmetric and not in $F(s)$.

2. **A new set of conditions.** In this section we shall greatly simplify the conditions of my Palermo paper that a field $F(x)$ shall be the centrum of D. Let ω be a Riemann matrix of genus p over a real algebraic field of finite degree F and let s be a multiplication of ω. Let $\sigma_1, \cdots, \sigma_t$ be the complex roots of the minimum equation of s for F. A necessary condition that ω be a pure Riemann matrix with s as a symmetric multiplication is that the above equation be irreducible in F, the numbers $\sigma_1, \cdots, \sigma_t$ be all real, p be divisible by t, $p = tp'$, and ω be isomorphic[14] in F to

$$(2) \qquad || \tau_j \sigma_j^{k-1} || \qquad (j, k = 1, \cdots, t).$$

Let V be the generalized Vandermonde matrix

$$(3) \qquad V = || \sigma_j^{k-1} I(2p') || \qquad (j, k = 1, \cdots, t),$$

[13] Except for the above values of h and k for algebras of the first kind the results of this section were obtained in a far shorter and simpler argument (than Rosati's original work) in my Palermo Rendiconti paper Section 3.

[14] We of course use the ordinary definition of isomorphism of Riemann matrices except that now G has elements in F instead of in R as in our introduction. For proof of the above necessary conditions on s, see my Palermo paper Section 4.

a non-singular matrix. Then the multiplication s of ω is given[15] by

$$s: \ \sigma\omega = \omega S, \tag{4}$$

where

$$\sigma = \begin{Vmatrix} \sigma_1 I(p') & & & 0 \\ & \sigma_2 I(p') & & \\ & & \cdot & \\ & & & \cdot \\ & & & \\ 0 & & & \sigma_t I(p') \end{Vmatrix}, \tag{5}$$

and S has elements in F and is such that

$$VSV^{-1} = \begin{Vmatrix} \sigma_1 I(2p') & & & 0 \\ & \sigma_2 I(2p') & & \\ & & \cdot & \\ & & & \cdot \\ & & & \\ 0 & & & \sigma_t I(2p') \end{Vmatrix}. \tag{6}$$

Evidently $\omega = \omega_0 V$, where

$$\omega_0 = \begin{Vmatrix} \tau_1 & & & 0 \\ & \tau_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & \\ 0 & & & \tau_t \end{Vmatrix}. \tag{7}$$

The matrix τ_j is a Riemann matrix over $F(\sigma_j)$ and has, as principal $F(\sigma_j)$-matrix, an alternate matrix $C(\sigma_j)$ whose form is independent of j.

Conversely, let τ_j be a Riemann matrix over $F(\sigma_j)$ with $C_j = C(\sigma_j)$ as principal matrix. Then ω in the form (2) is a Riemann matrix over F with the multiplication s and with C as a principal F-matrix[16] where C is uniquely determined by

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[15] These properties are due to S. Lefschetz, as is also the canonical form (2)–(7). They will be found in his memoir entitled Invariant numbers of algebraic varieties, Transactions of the American Mathematical Society, **22** (1921), pp. 327–482, p. 420. For a simple algebraic proof see my Rendiconti paper (loc. cit.) p. 142.

[16] A Riemann matrix over F may also be a Riemann matrix over some other field K. For example every Riemann matrix over F is isomorphic in F to a Riemann matrix over R. Hence the use of principal F-matrix is justified. However there will be no confusion if we omit the F henceforth as we give the field by saying that ω is a Riemann matrix over F with principal matrix (F-matrix) C.

$$(8) \qquad VCV' = \left\| \begin{array}{ccc} \cdot & & \\ & \cdot & \\ & & \cdot \\ & C(\sigma_j) & \\ & & \cdot \\ & & \cdot \\ & & \cdot \end{array} \right\| .$$

We seek necessary and sufficient conditions on the τ_j that all multiplications of ω be commutative with s.

Let G be a $2p$-rowed square matrix with elements in F. It is easily shown that

$$(9) \qquad VGV^{-1} = \| G(\sigma_j, \sigma_k) \| \qquad (j, k = 1, \cdots, t)$$

is a t-rowed square matrix whose element in the j^{th} row and k^{th} column is a polynomial in σ_j, σ_k with coefficients which are $2p'$-rowed square matrices with elements in F independent of j and k. Evidently

$$V(SG - GS)V^{-1} = \| (\sigma_j - \sigma_k)G(\sigma_j, \sigma_k) \|$$

is a zero matrix if and only if

$$(10) \qquad G(\sigma_k, \sigma_k) = 0 \qquad (j \neq k; j, k = 1, \cdots, t),$$

since $\sigma_j - \sigma_k = 0$ only when $j = k$ (when the minimum equation of s is irreducible in F).

Let G be a projectivity of ω so that $\rho\omega = \omega G$ for a complex p-rowed square matrix ρ. From (6) we deduce that $\rho\omega_0 = \omega_0 VGV^{-1}$. Write

$$\rho = \| \rho_{jk} \| \qquad (j, k = 1, \cdots, t).$$

Then $\rho\omega_0 = \omega_0 VGV^{-1}$ is equivalent to

$$(11) \qquad \rho_{jk}\tau_k = \tau_j G(\sigma_j, \sigma_k) \qquad (j, k = 1, \cdots, t).$$

Hence, if it is true that the system of matrix equations (11) cannot be satisfied unless

$$G(\sigma_j, \sigma_k) = 0 \qquad (j \neq k; j, k = 1, \cdots, t),$$

it will follow that every projectivity G of ω is commutative with S and hence that every multiplication of ω is commutative with s.

Conversely, let every multiplication of ω be commutative with s so that $GS = SG$ for every projectivity G of ω. Suppose that (11) is satisfied for some $G(\sigma_j, \sigma_k)$ which is not zero for at least one $j \neq k$. If $j \neq k$ the number $\sigma_j - \sigma_k \neq 0$ is in $F(\sigma_j, \sigma_k)$ and has an inverse $P(\sigma_j, \sigma_k)$ in this field. Write

$$(\sigma_j - \sigma_k)^{-1}G(\sigma_j, \sigma_k) = P(\sigma_j, \sigma_k)G(\sigma_j, \sigma_k) \equiv H(\sigma_j, \sigma_k).$$

Then $H(\sigma_j, \sigma_k)$ is a polynomial in σ_j and σ_k with coefficients which are $2p'$-

rowed square matrices with elements in F. For any value of j we define $H(\sigma_j, \sigma_j)$ to be the polynomial obtained by formally replacing σ_k by σ_j. Let now

$$R(\sigma_j, \sigma_k) \equiv (\sigma_j - \sigma_k)H(\sigma_j, \sigma_k) \qquad (j, k = 1, \cdots, t).$$

$R(\sigma_j, \sigma_k)$ is now completely defined. Evidently $R(\sigma_j, \sigma_k)$ vanishes for $j = k$, while $R(\sigma_j, \sigma_k) = G(\sigma_j, \sigma_k)$ for $j \neq k$. But then if $\rho_{jj} = 0$ we have

$$\rho_{jk}\tau_k = \tau_j R(\sigma_j, \sigma_k) \qquad (j, k = 1, \cdots, t), \tag{12}$$

and the corresponding multiplication of ω is not commutative with s, a contradiction of our initial hypotheses. We have proved

THEOREM 1. *The matrix ω of* (2) *has the property that every multiplication of ω is commutative with s if and only if the system of matrix equations* (11) *are not simultaneously solvable for $G(\sigma_j, \sigma_k)$ except for the trivial solution where $G(\sigma_j, \sigma_k) = 0$ for all $j \neq k$.*

For our existence theorems we shall obtain a far simpler sufficient condition as a consequence of the necessary and sufficient condition of Theorem 1.

Suppose that for a given set of τ_j we have shown that (11) cannot be solved simultaneously for $j = 1; k = 2, \cdots, t$, and with a $G(\sigma_1, \sigma_k) \neq 0$. We propose to show that every multiplication of ω is commutative with s. Let G be a projectivity of ω not commutative with S. By Theorem 1 the set of equations (11) are solvable. By hypothesis $G(\sigma_1, \sigma_k) = 0$ for $k = 2, \cdots, t$, but there exists a pair of integers $j = q, k = r$ for which $G(\sigma_j, \sigma_k) \neq 0$. Of course $q \neq 1$. The minimum equation of s is irreducible in F and hence has a transitive group so that there exists a substitution in this group which carries σ_q to σ_1. Designate this substitution by Θ and let Θ carry r to, let us say, d. Then Θ^{-1} carries σ_1 to σ_q and σ_d to σ_r. But $G(\sigma_1, \sigma_d) = 0$. Hence its elements are in F. By the definition of the Galois group of an equation the elements of $G(\sigma_1, \sigma_d)$ are unaltered by the substitution Θ^{-1}. Therefore $G(\sigma_q, \sigma_r) = 0$, contrary to the assertion that $G(\sigma_j, \sigma_k) \neq 0$. We have proved

THEOREM 2. *If the system of matrix equations*

$$\rho_{1k}\tau_k = \tau_1 G(\sigma_1, \sigma_k) \qquad (k = 2, \cdots, t) \tag{13}$$

is not simultaneously solvable unless $G(\sigma_1, \sigma_k) = 0$, then the Riemann matrix (2) *has no multiplication not commutative with s.*

In our further considerations the centrum of algebra D will usually be $F(s)$, but this will not be true for matrices of the second kind. We must therefore examine this case separately.

3. **Conditions for matrices of the second kind.** Assume now that s is not necessarily commutative with all the multiplications of ω but only that there exists a multiplication u of ω such that $us = su$, $u^2 = \mu(s)$ in $F(s)$ such that $\mu(\sigma_j) < 0$ $(j = 1, \cdots, t)$. Then ω has the form (2), where now

$$\tau_j = \begin{Vmatrix} \tau_{j1} & \tau_{j1}\,\mu_j^{\frac{1}{2}} \\ \tau_{j2} & -\tau_{j2}\,\mu_j^{\frac{1}{2}} \end{Vmatrix}, \tag{14}$$

with $\mu_j = \mu(\sigma_j)$. The matrix τ_{j1} has r_{j1} rows and p' columns while τ_{j2} has r_{j2} rows and p' columns, where $r_{j1} + r_{j2} = p'$. We may write

$$W_j = \begin{Vmatrix} I(p') & \mu_j^{\frac{1}{2}} I(p') \\ I(p') & -\mu_j^{\frac{1}{2}} I(p') \end{Vmatrix}, \tag{15}$$

and then see that

$$\tau_j = \begin{Vmatrix} \tau_{j1} & 0 \\ 0 & \tau_{j2} \end{Vmatrix} W_j. \tag{16}$$

Since τ_j is a Riemann matrix over $F(\sigma_j)$ with $C_j = C(\sigma_j)$ as principal matrix,

$$\begin{Vmatrix} \tau_{j1} & 0 \\ 0 & \tau_{j2} \end{Vmatrix} W_j C_j W_j' \begin{Vmatrix} \tau_{j1}' & 0 \\ 0 & \tau_{j2}' \end{Vmatrix} = 0, \tag{17}$$

and

$$\sqrt{-1} \begin{Vmatrix} \tau_{j1} & 0 \\ 0 & \tau_{j2} \end{Vmatrix} W_j C_j \bar{W}_j' \begin{Vmatrix} \bar{\tau}_{j1}' & 0 \\ 0 & \bar{\tau}_{j2}' \end{Vmatrix}. \tag{18}$$

is a positive definite Hermitian matrix. We may write

$$W_j C_j W_j' = \begin{Vmatrix} \Gamma(\mu_j^{\frac{1}{2}}, \mu_j^{\frac{1}{2}}) & \Gamma(\mu_j^{\frac{1}{2}}, -\mu_j^{\frac{1}{2}}) \\ \Gamma(-\mu_j^{\frac{1}{2}}, \mu_j^{\frac{1}{2}}) & \Gamma(-\mu_j^{\frac{1}{2}}, -\mu_j^{\frac{1}{2}}) \end{Vmatrix}, \tag{19}$$

and we shall then obtain

$$\tau_{j1}\Gamma(\mu_j^{\frac{1}{2}}, -\mu_j^{\frac{1}{2}})\tau_{j2}' = 0, \tag{20}$$

while at least we must have principal minors of (18) positive definite so that

$$i\tau_{j1}\Gamma(\mu_j^{\frac{1}{2}}, -\mu_j^{\frac{1}{2}})\bar{\tau}_j', \quad i\tau_{j2}\Gamma(-\mu_j^{\frac{1}{2}}, \mu_j^{\frac{1}{2}})\bar{\tau}_{j2}' \tag{21}$$

must be positive definite Hermitian matrices.

Conversely if C_j are so chosen that (20), (21) hold with

$$W_j C_j W_j' = \begin{Vmatrix} 0 & \Gamma(\mu_j^{\frac{1}{2}}) \\ \Gamma(-\mu_j^{\frac{1}{2}}) & 0 \end{Vmatrix}, \tag{22}$$

where $\Gamma(\mu_j^{\frac{1}{2}}) = \Gamma(\mu_j^{\frac{1}{2}}, -\mu_j^{\frac{1}{2}})$, then τ_j is a Riemann matrix over $F(\sigma_j)$ with C_j as principal matrix. We thus need consider only (22) and make

$$\tau_{j1}\Gamma(\mu_j^{\frac{1}{2}})\tau_{j2}' = 0, \quad i\tau_{j1}\Gamma(\mu_j^{\frac{1}{2}})\bar{\tau}_{j1}', \quad i\tau_{j2}\Gamma(-\mu_j^{\frac{1}{2}})\bar{\tau}_{j2}' \tag{23}$$

positive definite Hermitian matrices.

The matrix $\Gamma_j = \Gamma(\mu_j^{\frac{1}{2}})$ is a skew-Hermitian matrix when C_j is an alternate matrix, since $\mu_j^{\frac{1}{2}} = -\mu_j^{\frac{1}{2}}$, $\bar{\Gamma}_j = \Gamma(-\mu_j^{\frac{1}{2}})$, and

$$(W_j C_j W_j')' = \begin{Vmatrix} 0 & \bar{\Gamma}_j' \\ \Gamma_j' & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & -\Gamma_j \\ -\bar{\Gamma}_j & 0 \end{Vmatrix}.$$

Hence there necessarily exists a non-singular p-rowed square matrix G_j with elements in $F(\sigma_j, \mu_j^{\frac{1}{2}})$ such that

$$G_j \Gamma_j \bar{G}_j' = \mu_j^{\frac{1}{2}} H_j,$$

where H_j is a diagonal matrix with elements in $F(\sigma_j)$. But then if

$$\tau_{j3} = \tau_{j1} G_j^{-1}, \quad \tau_{j4} = \tau_{j2} \bar{G}_j^{-1},$$

we have

$$\tau_{j3} H_j \mu_j^{\frac{1}{2}} \tau_{j4}' = \tau_{j1} G_j^{-1} G_j \Gamma_j \bar{G}_j^{1} (\bar{G}_j^{1})^{-1} \tau_{j2}' = 0,$$

and

$$i\tau_{j3} H_j \mu_j^{\frac{1}{2}} \bar{\tau}_{j3}' = i\tau_{j1} G_j^{-1} G_j \Gamma_j \bar{G}_j^{1} (\bar{G}_j^{1})^{-1} \bar{\tau}_{j1}' = i\tau_{j1} \Gamma_j \bar{\tau}_j',$$
$$i\tau_{j4} \overline{(H_j \mu_j^{\frac{1}{2}})} \bar{\tau}_{j4}' = i\tau_{j2} G_j^{-1} G_j \Gamma_j \bar{G}_j^{1} (\bar{G}_j^{1})^{-1} \bar{\tau}_{j2}' = i\tau_{j2} \bar{\Gamma}_j \bar{\tau}_{j2}'$$

are positive definite Hermitian matrices. There exists a matrix E_j with elements in $F(\sigma_j)$ such that

$$W_j E_j W_j^{-1} = \left\| \begin{matrix} G_j & 0 \\ 0 & \bar{G}_j \end{matrix} \right\|,$$

and thus such that

$$\omega_j \equiv \tau_j E_j = \left\| \begin{matrix} \tau_{j1} & 0 \\ 0 & \tau_{j2} \end{matrix} \right\| W_j E_j W_j^{-1} W_j = \left\| \begin{matrix} \tau_{j3} & 0 \\ 0 & \tau_{j4} \end{matrix} \right\| W_j.$$

Also there exists a matrix E with elements in F such that

$$VEV^{-1} = \left\| \begin{matrix} E_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & E_t \end{matrix} \right\|,$$

so that E is non-singular and

$$\omega E = \left\| \begin{matrix} \tau_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \tau_t \end{matrix} \right\| (VEV^{-1})\, V = \left\| \begin{matrix} \omega_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \omega_t \end{matrix} \right\| V.$$

It follows that, by passing to a matrix isomorphic to ω we can take

$$\Gamma_j = \mu_j^{\frac{1}{2}} H_j \tag{24}$$

where H_j is a diagonal matrix with elements in $F(\sigma_j)$.

We shall now consider the important question as to what are possible values of the integers r_{j1} and r_{j2} when $r_{j1} = r_1, r_{j2} = r_2$. Suppose that $r_1 = 0$. Then

$$\tau_j = \| \tau_{j2} 0 \| \bar{W}_j, \tag{25}$$

and, by replacing W_j by $\bar{W}_j$ which has the same form, but with $\mu_j^{\frac{1}{2}}$ replaced by its complex conjugate $-\mu_j^{\frac{1}{2}}$, we can take τ_j without loss of generality to have the form

$$\tau_j = \| \tau_{j1}\, 0 \| \, W_j. \tag{26}$$

This is a τ_j matrix with $r_2 = 0$. Hence if $r_1 r_2 = 0$ we need consider only the case $r_2 = 0$. We shall be interested merely in the case where $p' > 1$. Let β be a p'-rowed square matrix whose element in the first row and first column is unity and whose other elements are all zero so that β has zero determinant. Evidently there exists a matrix B_j such that

$$W B_j W_j' = \begin{Vmatrix} 0 & \beta \\ \beta & 0 \end{Vmatrix} \tag{27}$$

since $\beta = \beta(\mu_j^{\frac{1}{2}}) = \beta(-\mu_j^{\frac{1}{2}})$. But

$$\tau_j B_j \tau_j' = \begin{Vmatrix} \tau_{j1} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & \beta \\ \beta & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \tau_{j1}' \\ 0 \end{Vmatrix} = 0. \tag{28}$$

It follows that the matrix

$$G_j = B_j \Gamma_j^{-1} \tag{29}$$

is a projectivity of τ_j and if

$$VGV^{-1} = \begin{Vmatrix} G_1 & & & 0 \\ & G_2 & & \\ & & \ddots & \\ 0 & & & G_t \end{Vmatrix}, \tag{30}$$

that G is a projectivity of ω. Evidently G has zero determinant. This is impossible when ω is pure.

THEOREM 3. *Let ω be a pure Riemann matrix of genus p over F and let ω have a symmetric multiplication s of grade t and a skew symmetric multiplication q commutative with s and with q^2 in $F(s)$ and integers $r_{j1} = r_1$, $r_{j2} = r_2$. Then $p = tp'$, p' an integer, and if $p' > 1$ the integers r_1 and r_2 are both not zero.*

Consider now the special case $p' = 2$ so that, since $r_1 + r_2 = 2$ and $r_1 r_2 \neq 0$, we have $r_1 = r_2 = 1$. We write

$$\tau_{j1} = \| \lambda_{j1}, \lambda_{j2} \|, \; \tau_{j2} = \| \lambda_{j3}, \lambda_{j4} \|. \tag{31}$$

Also if $F_j = F(\sigma_j)$ then, by (24), we may take

$$\Gamma_j = \begin{Vmatrix} \alpha_{j1} & 0 \\ 0 & \alpha_{j2} \end{Vmatrix} \mu_j^{\frac{1}{2}}, \quad \alpha_{j1}, \alpha_{2j} \text{ in } F_j, \tag{32}$$

where Γ_j is a non-singular matrix such that $\tau_{j1}\Gamma_j\tau_{j2}' = 0$ whence

$$\lambda_{j1}\lambda_{j3}\alpha_{j1} + \lambda_{j2}\lambda_{j4}\alpha_{j2} = 0. \tag{33}$$

We shall prove that

$$B_j = B(\sigma_j) = \begin{Vmatrix} \beta_j & 0 \\ 0 & -\beta_j \end{Vmatrix}, \quad \beta_j = \begin{Vmatrix} 0 & -\alpha_{j1} \\ \alpha_{j2} & 0 \end{Vmatrix}, \tag{34}$$

is a projectivity of τ_j.

We take

$$\rho_j = \begin{Vmatrix} 0 & \rho_{j1} \\ \rho_{j2} & 0 \end{Vmatrix}, \tag{35}$$

where ρ_{j1}, ρ_{j2} are numbers to be selected, and we propose to demonstrate that for properly chosen ρ_{j1}, ρ_{j2},

$$\rho_j\tau_j = \tau_jB_j. \tag{36}$$

Now

$$\begin{Vmatrix} 0 & \rho_{j1} \\ \rho_{j2} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \tau_{j1} & \tau_{j1}\mu_j^{\frac{1}{2}} \\ \tau_{j2} & -\tau_{j2}\mu_j^{\frac{1}{2}} \end{Vmatrix} = \begin{Vmatrix} \rho_{j1}\tau_{j2} & -\rho_{j1}\tau_{j2}\mu_j^{\frac{1}{2}} \\ \rho_{j2}\tau_{j1} & \rho_{j2}\tau_{j1}\mu_j^{\frac{1}{2}} \end{Vmatrix}, \tag{37}$$

while

$$\begin{Vmatrix} \tau_{j1} & \tau_{j1}\mu_j^{\frac{1}{2}} \\ \tau_{j2} & -\tau_{j2}\mu_j^{\frac{1}{2}} \end{Vmatrix} \cdot \begin{Vmatrix} \beta_j & 0 \\ 0 & -\beta_j \end{Vmatrix} = \begin{Vmatrix} \tau_{j1}\beta_j & -\tau_{j1}\beta_j\mu_j^{\frac{1}{2}} \\ \tau_{j2}\beta_j & \tau_{j2}\beta_j\mu_j^{\frac{1}{2}} \end{Vmatrix}. \tag{38}$$

Hence $\rho_j\tau_j = \tau_jB_j$ if and only if

$$\rho_{j1}\tau_{j2} = \tau_{j1}\beta_j, \quad \rho_{j2}\tau_{j1} = \tau_{j2}\beta_j. \tag{39}$$

But these equations are equivalent to

$$\rho_{j1}\lambda_{j3} = \lambda_{j2}\alpha_{j2}, \quad \rho_{j1}\lambda_{j4} = -\lambda_{j1}\alpha_{j1}, \quad \rho_{j2}\lambda_{j1} = \lambda_{j4}\alpha_{j2}, \quad \rho_{j2}\lambda_{j2} = -\lambda_{j3}\alpha_{j1}. \tag{40}$$

If $\lambda_{j3} = 0$ then, since τ_j is a Riemann matrix and has no row of zero elements, $\lambda_{j4} \neq 0$ and (33) gives $\lambda_{j2} = 0$, $\lambda_{j1} \neq 0$. But then (40) are satisfied when we take $\rho_{j1} = -\lambda_{j1}\alpha_{j1}\lambda_{j4}^{-1}$ and $\rho_{j2} = -\lambda_{j4}\alpha_{j2}\lambda_{j1}^{-1}$.

Next let $\lambda_{j3} \neq 0$ so that $\lambda_{j2} \neq 0$. Take

$$\rho_{j1} = \alpha_{j2}\lambda_{j2}\lambda_{j3}^{-1}, \quad \rho_{j2} = -\lambda_{j3}\alpha_{j1}\lambda_{j2}^{-1}. \tag{41}$$

Using (33) we have

$$\rho_{j1}\lambda_{j4} = \lambda_{j2}\alpha_{j2}\lambda_{j4}\lambda_{j3}^{-1} = -\lambda_{j1}\lambda_{j3}\alpha_{j1}\lambda_{j3}^{-1} = -\lambda_{j1}\alpha_{j1}, \tag{42}$$

$$\rho_{j2}\lambda_{j1} = -\lambda_{j3}\lambda_{j1}\alpha_{j1}\lambda_{j2}^{-1} = \lambda_{j2}\alpha_{j2}\lambda_{j4}\lambda_{j2}^{-1} = \lambda_{j4}\alpha_{j2}, \tag{43}$$

as desired. We have proved that in all cases we can choose ρ_j such that $\rho_j\tau_j = \tau_j B_j$.

Let

$$Q_j = \begin{Vmatrix} 0 & \mu_j I(p') \\ I(p') & 0 \end{Vmatrix}. \tag{44}$$

Then

$$Q_j B_j = \begin{Vmatrix} 0 & -\mu_j \beta_j \\ \beta_j & 0 \end{Vmatrix}, \quad B_j Q_j = \begin{Vmatrix} 0 & \mu \beta_j \\ -\beta_j & 0 \end{Vmatrix}. \tag{45}$$

Hence $Q_j B_j = -B_j Q_j$. Define a matrix B by

$$VBV^{-1} = \begin{Vmatrix} B_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & B_t \end{Vmatrix}. \tag{46}$$

Evidently

$$QB = -BQ. \tag{47}$$

We have proved for the case $r_{j1} = r_1$, $r_{j2} = r_2$

THEOREM 4. *Let ω in the form (2) be a pure Riemann matrix with a multiplication q such that $qs = sq$, $q^2 = \mu(s)$ in $F(s)$, $\mu(\sigma_j) < 0$ $(j = 1, \cdots, t)$, and let $p = 2t$. Then ω has a multiplication b such that $bq = -qb$.*

II. EXISTENCE THEOREMS

1. **Elementary results.** We shall first prove

THEOREM 5. *Let K be any algebraic number field of finite degree, let m, e_1, $\cdots$, e_m, be any positive integers, and let*

$$0 \leq \delta_j < \epsilon_j \qquad (j = 1, \cdots, m) \tag{48}$$

for real δ_j, ϵ_j. Then there exist real algebraic numbers $\lambda_1, \cdots, \lambda_m$ chosen seriatim, so that δ_j, ϵ_j may be functions of $\lambda_1, \cdots, \lambda_{j-1}$ such that

$$0 \leq \delta_j < \lambda_j < \epsilon_j, \tag{49}$$

and the products

$$\lambda_1^{k_1}\lambda_2^{k_2} \cdots \lambda_m^{k_m} \quad (k_j = 0, 1, \cdots, e_j - 1; j = 1, \cdots, m), \tag{50}$$

are linearly independent with respect to K.

We first let K be real, $m = 1$, e a positive odd integer, and $0 \leq \delta < \epsilon$, where δ and ϵ are arbitrary real numbers with $\delta \geqq 0$, $\epsilon > 0$. By the well known

Hilbert Irreducibility Theorem there exists an equation $g(x) = 0$ of degree e with coefficients in K and irreducible in K. But $g(x)$ has odd degree and a real root ξ, an algebraic number. Evidently the condition that $g(x)$ is irreducible implies that $1, \xi, \xi^2, \cdots, \xi^{e-1}$ are linearly independent with respect to K and, if $\lambda = \eta\xi$ is any rational multiple of ξ then $1, \lambda, \lambda^2, \cdots, \lambda^{e-1}$ are linearly independent with respect to K. But evidently η may be chosen so that $0 \leq \delta < \eta\xi < \epsilon$. Hence we have proved Theorem 4 for the case of a real K, $m = 1$, $\epsilon_j = \epsilon_1$ odd.

We next consider the case where K is still real but otherwise the hypotheses of Theorem 5 are all satisfied. Let $t_j = e_j$ or $e_j + 1$ according as e_j is odd or even. By our case $m = 1$ there exists a real algebraic number λ_1 such that $\delta_1 < \lambda_1 < \epsilon_1$ and $K_1 = K(\lambda_1)$ is real and has order t_1 with respect to $K_0 = K$. Similarly there exists a real algebraic number λ_2 such that $\delta_2 < \lambda_2 < \epsilon_2$ and the algebraic field $K_2 = K_1(\lambda_2)$ has order t_2 with respect to K_1. In general, if K_{j-1} has been chosen, then there exists a real algebraic number field $K_j = K_{j-1}(\lambda_j)$ of order t_j with respect to K_{j-1} $(j = 1, \cdots, m)$. The field $J = K(\lambda_1, \cdots, \lambda_m) = K_m$ has order

$$t = t_1 \cdot \cdots \cdot t_m \tag{51}$$

with respect to K and a basis

$$\lambda_1^{k_1} \cdot \cdots \cdot \lambda_m^{k_m} \quad (k_j = 0, 1, \cdots, t_j - 1; j = 1, \cdots, m). \tag{52}$$

Since (52) are linearly independent with respect to K so also are the abridged set (50). This completes the case K real.

Let now $K = R(\theta)$ where θ is an imaginary root of $g(x) = 0$ with rational coefficients and irreducible in R. Then $g(\bar{\theta}) = 0$, where $\bar{\theta}$ is the complex conjugate of θ so that if

$$\alpha = \theta + \bar{\theta}, \beta = \theta - \bar{\theta}, \tag{53}$$

then α and β are algebraic numbers. Evidently if

$$K_0 = R(\alpha, \beta^2), S = R(\theta, \bar{\theta}), \tag{54}$$

then $S = K_0(\beta)$. Also K_0 is a real field and we may apply Theorem 5 to the case already treated to obtain quantities λ_j satisfying all the desired conclusions except that now (50) have been proved linearly independent with respect merely to K_0 and not K. But the λ_j are all real. We can then prove the products (50) actually linearly independent with respect to S and hence its sub-field K.

For, let a linear combination with coefficients in S not all zero of (50) be zero. Since $S = K_0(\beta)$ we may write our linear combination in the form $A + B\beta = 0$ where A and B are linear combinations, with coefficients in K_0, of the quantities (50). These coefficients are all zero if and only if the original coefficients in S are all zero. By hypothesis A and B are not both zero since their coefficients in K_0 are not all zero and the quantities (50) are linearly independent in K_0. But (50) are real, K_0 is real, while then A and B are

real. But β is pure imaginary. Hence $A + B\beta = 0$ implies, by passing to complex conjugates, $A - B\beta = 0$ whence $A = B = 0$. This contradicts the independency of (50). This completes the proof of Theorem 5.

We next obtain a further elementary theorem. First we prove

LEMMA. *Let*

$$E = \| \epsilon_{jk} \| \qquad (j, k = 1, \cdots, m), \tag{55}$$

be a complex m-rowed Hermitian matrix with prescribed ϵ_{jk} for $j \neq k$ but with arbitrary ϵ_{jj}. Let ϵ be any real number such that

$$\epsilon \geq | \epsilon_{jk} | \tag{56}$$

(where $|\epsilon_{jk}|$ is the absolute value of ϵ_{jk}). Then if we select the ϵ_{jj} to be positive real numbers satisfying the inequality

$$\epsilon_{jj} > (m! - 1)\epsilon, \tag{57}$$

the matrix E has positive determinant.

For proof notice that for $m = 1$ the lemma is trivially true. Let then $m > 1$ so that $m! - 1 \geq 1$. Hence each $\epsilon_{jj} > \epsilon$.

The expansion of the determinant of E is a sum

$$\epsilon_{11}\epsilon_{22} \cdot \cdots \cdot \epsilon_{mm} + \Delta, \tag{58}$$

where Δ is a real sum of $m! - 1$ complex numbers. Let δ be the smallest of the ϵ_{jj} and define γ by

$$\epsilon_{11}\epsilon_{22} \cdot \cdots \cdot \epsilon_{mm} = \delta\gamma. \tag{59}$$

It evidently suffices to prove that $\gamma\delta > |\Delta|$.

Every term of Δ has at least one factor $\epsilon_{jk} (j \neq k)$ and the product of the remaining factors is obviously at most γ in absolute value since each $|\epsilon_{jk}| \leq \epsilon$, $\epsilon_{jj} \geq \delta$, $\epsilon_{jj} > \epsilon$. Hence every term of Δ has absolute value less than $\epsilon\gamma$. But then

$$| \Delta | \leq (m! - 1)\epsilon\gamma, \tag{60}$$

since Δ has $m! - 1$ such terms. Also $\delta > (m! - 1)\epsilon$,

$$\gamma\delta - (m! - 1)\gamma\epsilon = \gamma [\delta - (m! - 1)\epsilon] > 0, \gamma\delta > (m! - 1)\epsilon\gamma \geq | \Delta |, \tag{61}$$

as desired.

As a consequence of this lemma we have

THEOREM 6. *Let*

$$E = \| G_{jk} \|, \qquad (j, k = 1, \cdots, m), \tag{62}$$

be a complex m-rowed Hermitian matrix and let

$$\epsilon \geq | \epsilon_{jk} | \qquad (j \neq k).$$

Then if

$$\epsilon_{jj} > (m! - 1)\epsilon,$$

the matrix E is positive definite Hermitian.

For, a necessary and sufficient condition that E be positive definite is that every principal minor of E have positive determinant. Every r-rowed principal minor of E has the same form as E where now $\epsilon_{ij} > (m! - 1)\epsilon < (r! - 1)\epsilon$. Hence the above lemma implies that every principal minor of E has positive determinant as desired.

We next apply the above results to Riemann matrices.

2. **Riemann matrices over K.** Let K be a real algebraic number field of finite degree and let

$$\omega = \| I(p), \pi \|, \tag{63}$$

where $I(p)$ is the p-rowed identity matrix,

$$\pi = \pi_1 + \pi_2\sqrt{-1}, \tag{64}$$

and π_1 and π_2 are real symmetric p-rowed square matrices. We choose

$$C = \begin{Vmatrix} 0 & I(p) \\ -I(p) & 0 \end{Vmatrix} \tag{65}$$

and obtain

$$\omega C\omega' = \left\| I, \pi \right\| \cdot \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix} \cdot \begin{Vmatrix} I \\ \pi \end{Vmatrix} = -\pi + \pi = 0, \tag{66}$$

while, if $i = \sqrt{-1}$,

$$\omega iC\bar{\omega}' = \left\| I, \pi \right\| \cdot \begin{Vmatrix} 0 & iI \\ -iI & 0 \end{Vmatrix} \cdot \begin{Vmatrix} I \\ \bar{\pi} \end{Vmatrix} = -i\pi + i\bar{\pi} = 2\pi_2, \tag{67}$$

which is positive definite Hermitian if and only if π_2 is a positive definite symmetric matrix.

Consider the p-rowed square matrix

$$P = \begin{Vmatrix} 1 & 1 & . & . & . & 1 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 \end{Vmatrix}, \tag{68}$$

of determinant unity. The matrix

$$\pi_2 = PP' = \begin{Vmatrix} p & 1 & 1 & . & . & 1 \\ 1 & 1 & 0 & . & . & . \\ 1 & 0 & 1 & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & 1 \end{Vmatrix} \tag{69}$$

is evidently positive definite since, if P is any non-singular real matrix, the matrix PP' is positive definite symmetric. But π_1 is still at our choice.

Let π_1 be a diagonal matrix whose elements off the diagonal are all zero (so that π_1 is symmetric) and whose j^{th} diagonal element we designate by λ_j. Let λ_j be a real algebraic number chosen so that

$$1, \lambda_j, \lambda_j\lambda_k \ (j, k = 1, \cdots, p) \tag{70}$$

are linearly independent with respect to K. Such λ_j exist by Theorem 5 with $m = p$, $e_j = 2$. We now insert the well known

DEFINITION. *A Riemann matrix ω over K is said to possess a complex multiplication g if the projectivity G defining g is not a scalar matrix.*

Evidently the equation

$$\gamma\omega = \omega G \tag{71}$$

implies that G is a $2p$-rowed square matrix. Let the ω of (63) have a complex multiplication (71) and write

$$G = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}, \tag{72}$$

where a, b, c, d are p-rowed square matrices to which we give the notation

$$(73)\quad a = \| a_{jk} \|, b = \| b_{jk} \|, c = \| c_{jk} \|, d = \| d_{jk} \|, (j, k = 1, \cdots, p).$$

The elements a_{jk}, b_{jk}, c_{jk}, d_{jk} are evidently in K. Also write

$$\pi_2 = \| \pi_{jk} \| \qquad (j, k = 1, \cdots, p). \tag{74}$$

Equations (71) and (63) imply that $\gamma = a + \pi c$, $\gamma\pi = b + \pi d$ so that, since the matrices a, b, c, d are real,

$$a\pi_1 + \pi_1 c\pi_1 - \pi_2 c\pi_2 = b + \pi_1 d, \tag{75}$$

$$a\pi_2 + \pi_1 c\pi_2 - \pi_2 c\pi_1 = \pi_2 d. \tag{76}$$

The elements in the j^{th} row and k^{th} column of the members of (75) give

$$a_{jk}\lambda_k + \lambda_j c_{jk}\lambda_k - \Sigma\ \pi_{js}c_{st}\pi_{tk} - b_{jk} - \lambda_j d_{jk} = 0, \tag{77}$$

for $j, k = 1, \cdots, p$. By the linear independence of (70) in K we have

$$c_{jk} = 0, \quad b_{jk} = 0 \qquad (j, k = 1, \cdots, p), \tag{78}$$

so that $b = c = 0$. Also, using these values in (77) we obtain

$$0 = a_{jk} = d_{jk} \text{ for } j \neq k, \quad a_{jj} = d_j \ (j, k = 1, \cdots, p), \tag{79}$$

and a and d are equal diagonal matrices. Now use (76) which becomes

$$a\pi_2 = \pi_2 a, \tag{80}$$

since $b = c = 0$, $a = d$. Then

$$a_{jj}\pi_{jk} = \pi_{jk}a_{kk}. \tag{81}$$

Let $j = 1$ so that $\pi_{1k} \neq 0$ $(k = 1, \cdots, p)$ from (69). Then

$$a_{11}\pi_{1k} = \pi_{1k}a_{kk}, \quad a_{kk} = a_{11} \ (k = 2, \cdots, p), \tag{82}$$

and G is a scalar matrix. We have proved

THEOREM 7. *Let ω be given by* (63), (64), (69) *and*

$$\pi_1 = \begin{Vmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{Vmatrix}, \tag{83}$$

where the λ_j are real algebraic numbers chosen as in Theorem 5 so that

$$1, \lambda_j, \lambda_j\lambda_k \ (j, k = 1, \cdots, p) \tag{84}$$

are linearly independent with respect to K. Then ω is a pure Riemann matrix over K with (65) *as principal matrix and with no complex multiplication.*

We shall require the

DEFINITION. *Let ω_2 and ω_1 be two matrices with complex elements. Then we shall say that ω_2 is pseudoisomorphic to ω_1 in a field K if there exists a non-zero matrix G with elements in K and a complex matrix γ such that*

$$\gamma\omega_2 = \omega_1 G. \tag{85}$$

Let ω_1 and ω_2 be p-rowed and $2p$-columned matrices of the form

$$\omega_1 = \| \ I(p), \tau_1 \ \|, \ \omega_2 = \| \ I(p), \tau_2 \ \|, \tag{86}$$

where

$$\tau_1 = \pi_1 + \pi_2\sqrt{-1}, \ \tau_2 = \pi_3 + \pi_2\sqrt{-1}, \tag{87}$$

with π_1 and π_2 as in (83), (69) and with

$$\pi_3 = \begin{Vmatrix} \lambda_{p+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2p} \end{Vmatrix}. \tag{88}$$

Let the λ_j be chosen as in Theorem 3 such that

$$1, \lambda_j, \lambda_j\lambda_k \ (j, k = 1, \cdots, 2p) \tag{89}$$

are linearly independent with respect to a real algebraic field K containing two real algebraic fields K_1 and K_2. Since the elements of (65) are rational, the matrix ω_1 is a Riemann matrix over K_1, the matrix ω_2 is a Riemann matrix over K_2 (obviously for any K_1 and K_2). Moreover, by Theorem 7 each has no complex multiplication. We propose to show that ω_2 is not pseudoisomorphic to ω_1 in K. Let then

$$\gamma\omega_2 = \omega_1 G, \tag{90}$$

where G has elements in K not all zero and is necessarily a $2p$-rowed square matrix. We shall also write

$$G = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|, \tag{91}$$

as before where a, b, c, d are p-rowed square matrices with elements in K designated as in (73). Then

$$\gamma = a + \tau_1 c, \ \gamma\tau_2 = b + \tau_1 d,$$

so that

$$a\tau_2 + \tau_1 c\tau_2 = b + \tau_1 d, \tag{92}$$

and

$$a\pi_3 + \pi_1 c\pi_3 - \pi_2 c\pi_2 = b + \pi_1 d, \tag{93}$$

$$a\pi_2 + \pi_1 c\pi_2 - \pi_2 c\pi_3 = \pi_2 d, \tag{94}$$

since K is a real algebraic field. But then

$$a_{jk}\lambda_{p+k} + \lambda_j c_{jk}\lambda_{p+k} - \sum_{s,t} \pi_{js} c_{st} \pi_{tk} = b_{jk} + \lambda_j d_{jk}, \tag{95}$$

and, from the linear independence of the elements (89) in K

$$c_{jk} = 0, \ a_{jk} = d_{jk} = 0, \ b_{jk} = 0.$$

Hence $G = 0$, contrary to the definition of G in (90). We have proved

THEOREM 8. *Let K, K_1, K_2 be any real algebraic fields such that K has finite degree and contains K_1 and K_2. Suppose that ω_1 and ω_2 are given by* (86), (87), (83), (89), (88). *Then we may select the λ_j of* (88) *so that*

$$1, \lambda_j, \lambda_j\lambda_k \ (j, k = 1, \cdots, 2p)$$

are real algebraic numbers linearly independent with respect to K and obtain ω_i. The matrices ω_i are Riemann matrices over K_i with (85) *as principal matrix and with no complex multiplication, $(i = 1, 2)$, and ω_2 is not pseudoisomorphic to ω_1 in K.*

We have now obtained an existence theorem for Riemann matrices of the first kind. For Riemann matrices of the second kind we shall require the following further considerations.

3. **An existence theorem for matrices of the second kind.** Let $\mu < 0$ be in K and let q be a quantity such that $q^2 = \mu$. We propose to prove that there exist Riemann matrices of genus $p > 2$ with $K(q)$ as multiplication algebra.

Let

$$\Gamma = \left\| \begin{matrix} a & 0 \\ 0 & -1 \end{matrix} \right\| \tag{96}$$

be a p-rowed square matrix with ordinary integer elements. We take a to be the identity matrix of $p - 1$ rows. Take

$$\tau_1 = \| a, \pi \|, \tau_2 = \| -\pi', -1 \|, \tag{97}$$

where π is a one columned matrix of $p - 1$ real elements

$$\lambda_1, \cdots, \lambda_{p-1}. \tag{98}$$

Then τ_1 is a $(p\text{-}1)$-rowed and p-columned real matrix and τ_2 is a one-rowed and p-columned real matrix. Define[17]

$$\omega = \left\| \begin{matrix} \tau_1 & \tau_1 \mu^{\frac{1}{2}} \\ \tau_2 & -\tau_2 \mu^{\frac{1}{2}} \end{matrix} \right\|, \tag{99}$$

a matrix with p rows and $2p$ columns of complex elements. Also let

$$C = \left\| \begin{matrix} 0 & C_1 \\ -C_1 & 0 \end{matrix} \right\|, \quad C_1 = (-2\mu)^{-1}\Gamma, \tag{100}$$

so that C is a $2p$-rowed square alternate matrix with elements in K. The matrix ω is a Riemann matrix over K with C as principal matrix if and only if

$$\omega C \omega' = 0 \text{ and } \omega\sqrt{-1}C\bar{\omega}' \tag{101}$$

is a positive definite Hermitian matrix. But $\mu < 0$ so that these conditions are equivalent to the conditions that

$$\omega\mu^{\frac{1}{2}}C\omega' = 0 \text{ and } \mu^{\frac{1}{2}}\omega C\bar{\omega}' \tag{102}$$

be a positive definite Hermitian matrix. We compute

$$\omega\ \mu^{\frac{1}{2}}\ C\ \omega' = \left\| \begin{matrix} 0 & -2\mu\ \tau_1 C_1\ \tau_2' \\ 2\mu\ \tau_2 C_1\ \tau_1' & 0 \end{matrix} \right\| = \left\| \begin{matrix} 0 & \tau_1\ \Gamma\ \tau_2' \\ -\tau_2\ \Gamma\ \tau_1' & 0 \end{matrix} \right\|, \tag{103}$$

which is a zero matrix if and only if $\tau_1\Gamma\tau_2' = 0$ since $(\tau_1\Gamma\tau_2')' = \tau_2\Gamma\tau_1'$. But

$$\tau_1\ \Gamma\ \tau_2' = \left\| a, \pi \right\| \cdot \left\| \begin{matrix} a & 0 \\ 0 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} -\pi \\ -1 \end{matrix} \right\| = -\pi + \pi = 0, \tag{104}$$

since $a = I(p - 1)$. Also

$$\mu^{\frac{1}{2}}\ \omega\ C\ \bar{\omega}' = \left\| \begin{matrix} \tau_1\ \Gamma\ \tau_1' & 0 \\ 0 & -\tau_2\ \Gamma\ \tau_2' \end{matrix} \right\| \tag{105}$$

is a positive definite Hermitian matrix if and only if

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[17] We have evidently taken $r_1 = p - 1$, $r_2 = 1$ where r_1 and r_2 are the integers of theorem 3.

$$(106)\quad \tau_2 \Gamma \tau_2' = \left\| -\pi', -1 \right\| \cdot \left\| \begin{matrix} a & 0 \\ 0 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} -\pi \\ -1 \end{matrix} \right\| = \pi' a \pi - 1 = \sum_{j=1}^{p-1} \lambda_j^2 - 1 < 0,$$

and

$$(107)\quad \tau_1 \Gamma \tau_1' = \left\| a, \pi \right\| \cdot \left\| \begin{matrix} a & 0 \\ 0 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} a \\ \pi' \end{matrix} \right\| = a - \pi\pi'$$

is a positive definite symmetric matrix.

Let Λ be the $(p-1)$-rowed diagonal matrix whose j^{th} diagonal element is λ_j. Then

$$(108)\quad a - \pi\pi' = \Lambda P \Lambda',$$

where

$$(109)\quad P = \| P_{jk} \|, P_{jk} = -1 \text{ for } j \neq k, P_{jj} = \gamma_j = \lambda_j^{-2} - 1 \; (j, k = 1, \cdots, p).$$

Select the λ_j so that $0 < \lambda_j < [(p-1)!]^{-\frac{1}{2}}$. In this case

$$(110)\quad \lambda_j^2 < \frac{1}{(p-1)!}$$

and hence

$$(111)\quad \sum_{j=1}^{p-1} \lambda_j^2 < \frac{1}{(p-2)!} \leq 1$$

if $p > 2$. It is now obvious that $\tau_2 \Gamma \tau_2' < 0$ as desired. Also

$$(112)\quad \gamma_j = \lambda_j^{-2} - 1 > (p-1)! - 1,$$

so that, by Theorem 6 with $\epsilon = 1$, the matrix P is a positive definite symmetric matrix. It follows immediately that $\Lambda P \Lambda'$ is positive definite since Λ is non-singular by our choice of the λ_j.

THEOREM 9. *Let $p > 2$, $\mu < 0$ be in a real algebraic field K,*

$$(113)\quad \omega = \left\| \begin{matrix} \tau_1 & \tau_1 \mu^{\frac{1}{2}} \\ \tau_2 & -\tau \mu^{\frac{1}{2}} \end{matrix} \right\|,$$

where

$$(114)\quad \tau_1 = \| I(p-1), \pi \|, \tau_2 = \| -\pi^1, -1 \|,$$

and π has one column of $p-1$ elements λ_j. Let λ_j be chosen to be real algebraic numbers such that

$$(115)\quad \lambda_j^2 < \frac{1}{(p-1)!},$$

as in Theorem 6. *Then* ω *is a Riemann matrix over* K *with*

$$C = \begin{Vmatrix} 0 & C_1 \\ -C_1 & 0 \end{Vmatrix}, \quad -2\mu C_1 = \begin{Vmatrix} I(p-1) & 0 \\ 0 & -1 \end{Vmatrix}, \tag{116}$$

as principal matrix.

4. **Matrices not pseudoisomorphic.** We shall require an extension of Theorem 9 to the case of two Riemann matrices like the ω of Theorem 9. Moreover we shall discuss the multiplication algebra of each ω. First we may evidently write

$$\omega = \omega_0 W, \ \omega_0 = \begin{Vmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{Vmatrix}, \ W = \begin{Vmatrix} I(p) & \mu^{\frac{1}{2}} I(p) \\ I(p) & -\mu^{\frac{1}{2}} I(p) \end{Vmatrix}, \tag{117}$$

and then, if

$$Q = \begin{Vmatrix} 0 & \mu I(p) \\ I(p) & 0 \end{Vmatrix}, \tag{118}$$

we have

$$WQW' = \begin{Vmatrix} \mu^{\frac{1}{2}} I(p) & 0 \\ 0 & -\mu^{\frac{1}{2}} I(p) \end{Vmatrix}. \tag{119}$$

The matrix ω of (117) has the multiplication

$$q: \quad \begin{Vmatrix} \mu^{\frac{1}{2}} I(p-1) & 0 \\ 0 & -\mu^{\frac{1}{2}} \end{Vmatrix} \omega = \omega Q$$

since

$$\begin{Vmatrix} \mu^{\frac{1}{2}} I(p-1) & 0 \\ 0 & -\mu^{\frac{1}{2}} \end{Vmatrix} \begin{Vmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{Vmatrix} = \begin{Vmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{Vmatrix} WQW^{-1} = \begin{Vmatrix} \mu^{\frac{1}{2}} \tau_1 & 0 \\ 0 & -\mu^{\frac{1}{2}} \tau_2 \end{Vmatrix}. \tag{120}$$

I have proved[18] that ω has no multiplication except for those commutative with q if and only if there exists no non-zero matrix B with elements in $K(\mu^{\frac{1}{2}})$ such that

$$\xi_1 \tau_2 = \tau_1 B, \ \xi_2 \bar{\tau}_1 = \bar{\tau}_2 B, \tag{121}$$

for complex matrices ξ_1 and ξ_2. We may write

$$B = \begin{Vmatrix} P & R \\ S & t \end{Vmatrix}, \tag{122}$$

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[18] See my paper *On the structure of matrices*, etc. these Annals (loc. cit. *A*), p. 143–144.

where P is a $(p-1)$-rowed square matrix, R has $p-1$ rows and one column, S has one row and $p-1$ columns, and t is a number of $K(\mu^{\frac{1}{2}})$. Then the first equation of (121) becomes in our new notation

$$\xi_2 \,||\, -\pi', -1 \,||\, = ||\, I(p-1), \pi \,||\, B, \tag{123}$$

so that

$$\xi_1 = -(R + \pi t), \tag{124}$$

and hence

$$\pi'(R + \pi t)\,(P + \pi S) = 0. \tag{125}$$

Let the element in the j^{th} row and k^{th} column of P be designated by P_{jk}, the element in the j^{th} row of R by R_j, the element in the k^{th} column of S by S_k where $j, k = 1, 2, \cdots, p-1$. Then the element in the j^{th} row and k^{th} column of the left member of (125) is

$$\lambda_k R_j + \lambda_k \lambda_j t - P_{jk} - \lambda_j S_k = 0. \tag{126}$$

By Theorem 5 the λ_j can be chosen so as to satisfy (115) and yet such that

$$1, \lambda_j, \lambda_j\lambda_k \;(j, k = 1, \cdots, p-1) \tag{127}$$

are linearly independent in $K(\mu^{\frac{1}{2}})$. If $j \neq k$ equations (126) now imply that

$$t = P_{jk} = R_j = S_k = 0. \tag{128}$$

Also, if $j = k$, then $R_j = S_j$ $(j = 1, \cdots, p)$. But the value $j = 1$, $k = 2$ gives $R_1 = 0$ so that $S_1 = 0$, $R_j = S_1 = 0$ for $j = 2, \cdots, p$. Hence $S_j = 0$, $j = 1, \cdots, p$. Finally $B = 0$ and every multiplication of ω is commutative with q.

I have also shown (loc. cit.) that if every multiplication of ω is commutative with q the field $K(q)$ is the multiplication algebra of ω if and only if the only p-rowed square matrices B for which

$$\xi_1\tau_1 = \tau_1 B,\ \xi_2\tau_2 = \tau_2 B(-\mu^{\frac{1}{2}}) \tag{129}$$

are scalar matrices, where ξ_1 and ξ_2 are complex matrices, while B has elements in $K(\mu^{\frac{1}{2}})$. If we use the same notation as before, the equation (129) implies that

$$\xi_1 = P + \pi S,\ (P + \pi S)\pi - (R + \pi t) = 0, \tag{130}$$

and

$$\sum_{k=1}^{p-1} p_{jk}\lambda_k + \lambda_j \sum_k S_k \lambda_k - R_j - \lambda_j t = 0. \tag{131}$$

As before, since the P_{jk}, S_k, R_j, t are in $K(\mu^{\frac{1}{2}})$, equation (131) implies that S and R are zero matrices while

$$P_{jk} = 0 \text{ for } j \neq k,\ P_{jj} = t, \tag{132}$$

so that B is a scalar matrix as desired. Hence ω *has* $K(q)$ *as its multiplication algebra.* We now pass to the case of two matrices.

Let $\mu_1 < 0$ be in K_1 and $\mu_2 < 0$ be in K_2. Let

$$\omega_1 = \left\| \begin{matrix} \tau_1 & 0 \\ 0 & \tau_2 \end{matrix} \right\| W_1, \quad \omega_2 = \left\| \begin{matrix} \tau_3 & 0 \\ 0 & \tau_4 \end{matrix} \right\| W_2, \tag{133}$$

$$W_1 = \left\| \begin{matrix} I(p) & \mu_1^{\frac{1}{2}} I(p) \\ I(p) & -\mu_1^{\frac{1}{2}} I(p) \end{matrix} \right\|, \quad W_2 = \left\| \begin{matrix} I(p) & \mu_2^{\frac{1}{2}} I(p) \\ I(p) & -\mu_2^{\frac{1}{2}} I(p) \end{matrix} \right\|. \tag{134}$$

Let ω_1 be a matrix ω chosen as in the above considerations where we take K_1 and K_2 to be sub-fields of a real algebraic number field K which is the K for the linear independence of the quantities (127). But now we choose ω_2 in a similar way. For ω_2 of (133) we take

$$\tau_3 = \| I(p-1), \pi_2 \|, \tau_4 = \| -\pi_2', -1 \|, \tag{135}$$

such that if

$$\lambda_{p+j-1} = \nu_j \ (j = 1, \cdots, p-1), \tag{136}$$

then

$$\nu_j^2 < \frac{1}{(p-1)!}. \tag{137}$$

We replace the condition on (127) by the stronger condition that

$$1, \lambda_j, \lambda_j \lambda_k \ (j, k = 1, \cdots, 2p-2) \tag{138}$$

be linearly independent with respect to $K(\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}})$. This again can be accomplished by Theorem 5.

We have chosen ω_1 to be a pure Riemann matrix over K_1 with $K_1(q_1)$, $q_1^2 = \mu_1$ in K_1, as multiplication algebra with the corresponding result for ω_2. We now suppose that there exists a $2p$-rowed non-zero matrix G with elements in K such that

$$\gamma\omega_2 = \omega_1 G, \tag{139}$$

for a complex matrix γ. Then

$$\gamma \cdot \left\| \begin{matrix} \tau_3 & 0 \\ 0 & \tau_4 \end{matrix} \right\| = \left\| \begin{matrix} \tau_1 & 0 \\ 0 & \tau_2 \end{matrix} \right\| \cdot W_1 G W_2^{-1}. \tag{140}$$

By a simple computation

$$W_1 G W_2^{-1} = \left\| \begin{matrix} B(\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}) & B(\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}) \\ B(-\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}) & B(-\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}) \end{matrix} \right\| \tag{141}$$

so that if

$$\gamma = \left\| \begin{matrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{matrix} \right\|, \tag{142}$$

then

$$\gamma_1\tau_3 = \tau_1 B(\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}), \quad \gamma_2\tau_4 = \tau_1 B(\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}), \tag{143}$$

$$\gamma_3\tau_4 = \tau_2 B(-\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}), \quad \gamma_4\tau_4 = \tau_2 B(-\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}). \tag{144}$$

Consider the equations

$$\rho\tau_3 = \tau_1 B, \quad \sigma\tau_4 = \tau_1 B, \tag{145}$$

where B has the same notation as in our previous discussions of this section. The first equation of (143) gives

$$\rho = P + \pi S, \quad (P + \pi S)\pi_2 = R + \pi t, \tag{146}$$

so that

$$\sum_{k=1}^{p-1} P_{jk}\nu_k + \lambda_j \sum_k S_k\nu_k - R_j - \lambda_j t = 0. \tag{147}$$

But then

$$P = R = S = t = 0, \tag{148}$$

and B is a zero matrix. Next, the second equation of (143) is

$$\sigma\tau_4 = \sigma \,||\, -\pi_2', -1 \,|| = ||\, I(p-1), \pi \,||B, \tag{149}$$

so that

$$\sigma = -(R + \pi t), \; -\sigma\pi_2' = P + \pi S, \tag{150}$$

and

$$(R + \pi t)\pi_2' = P + \pi S. \tag{151}$$

Hence

$$R_j\nu_k + \lambda_j\nu_k t - (P_{jk} + \lambda_j S_k) = 0, \tag{152}$$

and again $B = 0$. Hence (143) is impossible unless $B(\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}) = B(\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}) = 0$. Passing to complex conjugates, $B(-\mu_1^{\frac{1}{2}}, -\mu_2^{\frac{1}{2}}) = B(-\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}) = 0$ since $\mu_1 < 0$, $\mu_2 < 0$. It follows that $WGW^{-1} = 0$ so that $G = 0$. We have proved

THEOREM 10. *Let $p > 2$, let K_1 and K_2 be real algebraic fields, and let K be a real algebraic field containing K_1 and K_2. Suppose that $\mu_1 < 0$ is in K_1, $\mu_2 < 0$ is in K_2, and*

$$\omega_1 = \left\| \begin{matrix} \tau_1 & \mu_1^{\frac{1}{2}}\tau_1 \\ \tau_2 & -\mu_1^{\frac{1}{2}}\tau_2 \end{matrix} \right\|, \quad \omega_2 = \left\| \begin{matrix} \tau_3 & \mu_2^{\frac{1}{2}}\tau_3 \\ \tau_4 & -\mu_2^{\frac{1}{2}}\tau_4 \end{matrix} \right\|, \tag{153}$$

where

$$(154)\quad \tau_1 = \| a, \pi_1 \|, \tau_2 = \| -\pi_1', -1 \|, \tau_3 = \| a, \pi_2 \|, \tau_4 = \| -\pi_2', -1 \|,$$

a is the p-rowed identity matrix, π_1 has one column composed of $p-1$ elements λ_j, and π_2 has one column composed of $p-1$ elements λ_{j+p-1} $(j = 1, \cdots, p-1)$. Let the λ_j be chosen to be real algebraic numbers so that

$$(155)\qquad \lambda_j^2 < \frac{1}{(p-1)!} \qquad (j = 1, \cdots, 2p-2)$$

and

$$(156)\qquad 1, \lambda_j, \lambda_j\lambda_k \ (j, k = 1, \cdots, 2p-2)$$

are linearly independent with respect to $K(\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}})$ as in Theorem 5. Then for $j = 1, 2$ the matrix ω_j is a pure Riemann matrix over K_j with multiplication algebra $K_j(q_j)$, $q_j^2 = \mu_j$, in K_j, with principal matrix

$$(157)\qquad C_j = \begin{Vmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{Vmatrix}, \ \Gamma = (-2\mu_j)^{-1} \begin{Vmatrix} I(p-1 & 0 \\ 0 & -1 \end{Vmatrix}, \ (j = 1, 2),$$

and ω_2 is not pseudoisomorphic to ω_1 in K.

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ON AN ALGEBRAIC GENERALIZATION OF THE QUANTUM MECHANICAL FORMALISM

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Introduction

One of us has shown that the statistical properties of the measurements of a quantum mechanical system assume their simplest form when expressed in terms of a certain hypercomplex algebra which is commutative but not associative.[1] This algebra differs from the non-commutative but associative matrix algebra usually considered in that one is concerned with the commutative expression $\frac{1}{2}(A \times B + B \times A)$ instead of the associative product $A \times B$ of two matrices. It was conjectured that the laws of this commutative algebra would form a suitable starting point for a generalization of the present quantum mechanical theory. The need of such a generalization arises from the (probably) fundamental difficulties resulting when one attempts to apply quantum mechanics to questions in relativistic and nuclear phenomena.

The mathematical foundations underlying this generalization have already been investigated completely[2] and the results obtained are summarized in Part I of this paper except for the following alterations: 1) We restrict ourselves entirely to commutative systems while in the investigation mentioned the commutative law was partly excluded. Further, it is of basic importance that we assume the "field of coefficients" of the algebra considered to be always real. (In this way we meet an old objection to the quantum mechanical formalism, namely, that it does not confine itself to the real domain. Complex numbers and the like are used here only as technical expedients.) 2) We give the unpublished proof of the deducibility of the law $(a^2b)a = a(ba^2)$ from the law $a^\mu a^\nu = a^{\mu+\nu}$. 3) We give determinative conditions for the further investigation in Part III of the structure of the "r-number systems"; these conditions appear in (23), (24), and (25) below.

Part II contains a summary discussion of the concepts "trace" and "interchangeability"; the investigation of these concepts has already been begun.[2]

The essential achievement of this paper is contained in Part III. There the structure of all (formally real, finite, and commutative) r-number systems is fully determined. It is shown that, with but one exception, all such systems are merely matrix algebras. This is noteworthy in itself from the mathematical point of view. But of more importance is the fact that the above mentioned

[1] P. Jordan, Zschr. f. Phys., vol. 80, 1933, p. 285.
[2] P. Jordan, Göttinger Nachr., 1932, p. 569; 1933, p. 209.

physical considerations demand an algebra essentially more general than matrix algebras. Our results therefore point to the necessity of further generalizations. Although we found *one* r-number system which is not a matrix algebra ($\mathfrak{M}_3^8$, cf. paragraph 22), this system is too narrow for the generalization needed. It may nevertheless be usable as a first step in this direction.

Apart from the possibility of dropping the requirement of commutativity, one first thinks of omitting the condition of finiteness, all the more so since the algebra appearing in quantum mechanics is, as well known, infinite (that is, it has no finite linear basis). It may well happen that new types of algebra will arise with the removal of some of the present restrictions. This possibility is further suggested by the fact that in the ordinary quantum mechanics many important features first appeared in terms of an infinite algebra.[3] But it is too early to make conjectures concerning these difficult questions.

However, our procedure can be regarded as a phenomenological basis for the matrix calculus of the quantum theory.

We wish to express here our appreciation to Dr. C. C. Torrance for his valuable assistance in connection with the preparation of this manuscript.

Foundations of the theory

1. We consider an algebra A with elements $a, b, c, \cdots$ such that from the elements a and b we can form the sum $a + b$, the difference $a - b$, and the product ab; also from the element a we can form the element λa, where λ is a real number. (In what follows, greek letters will always denote real numbers.) In our algebra all the usual rules of calculation are to hold (including the commutative law of multiplication) with the single exception of the associative law of multiplication.[4] Our algebra will have a finite linear base, that is, there will be a finite number of linearly independent elements $a_1, a_2, \cdots, a_N$ such that each element of the algebra is linearly dependent upon these a's. Further, our algebra will be "formally real" in the sense that

(I) If $a^2 + b^2 + c^2 + \cdots = 0$, then[5] $a = b = c = \cdots = 0$.

We call this algebra an r-number algebra if it has the properties

(II) $$a^\mu a^\nu = a^{\mu+\nu}, \qquad (\mu, \nu = 1, 2, \cdots)$$

(III) $$(a^2 b)a = a^2(ba),$$

---

[3] For example, the possibility of satisfying the commutation relations $pq - qp = \frac{h}{2\pi i} 1$, where h is a real number different from zero.

[4] That is, it is not necessary that $(ab)c = a(bc)$, though we desire that $(\lambda\mu)a = \lambda(\mu a)$ and that $(\lambda a)b = \lambda(ab)$.

[5] An algebra fulfilling this condition which, in a more general form, is better adapted to non-commutative algebras, has been called by Jordan[2] *semi-simple*. The present name, i.e., real, (used by E. Artin and I. Schreier, Hamb. Abh., vol. 5, 1926, p. 83) seems to be more suitable.

where a^μ is defined by

$$a^1 = a, \qquad a^{\mu+1} = a\,a^\mu \qquad (\mu = 1, 2, \cdots).$$

FUNDAMENTAL THEOREM 1. *According to the assumptions made above, including* (I), III *is a consequence of* II *and* II *is a consequence of* III.

The importance of this theorem lies in that (II) must be regarded as a necessary axiom because of the physical meaning of the theory,[6] whereupon the theorem shows that (III) is also a necessary law of any meaningful generalization of quantum mechanics.

The derivation of (III) from (II) is given at the end of paragraph 4. The derivation of (II) from (III) comes about in the following way:

We use the notation

$$[a, b, c] = (ab)c - a(bc). \tag{1}$$

From the commutative law it follows immediately that

$$[a, b, c] + [c, b, a] = 0, \tag{2}$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \tag{3}$$

If we replace b and a in (III) respectively by u and $\lambda a + \mu b + \nu c$, then an identity results in λ, μ, ν such that the terms proportional to $\lambda\mu\nu$ give

$$[ab, u, c] + [bc, u, a] + [ca, u, b] = 0. \tag{4}$$

Now suppose that it has already been shown that (II) follows from (III) for all μ and ν such that $\mu + \nu \leqq N$, where a^μ is determined by the definition given in (III). By (2) and (4) it follows that for $1 \leqq \nu \leqq N$

$$[a^{\nu+1}, b, a^{N-\nu-1}] = [a^\nu, b, a^{N-\nu}] + [a, b, a^{N-1}].$$

Hence

$$[a^\rho, b, a^{N-\rho}] = \rho[a, b, a^{N-1}] \text{ for } \rho = 1, 2, \cdots, N-1,$$

and therefore we have in particular that

$$-[a, b, a^{N-1}] = [a^{N-1}, b, a] = (N-1)\,[a, b, a^{N-1}],$$

$$[a, b, a^{N-1}] = 0, \qquad [a^\rho, b, a^{N-\rho}] = 0.$$

This gives the relation $a^{\rho+1}a^{N-\rho} = a^\rho a^{N-\rho+1} = a^1 a^N = a^{N+1}$ in the case where $b = a$. Hence (II) also holds for $\mu + \nu = N + 1$. Hence (II) holds in general and we have incidentally shown that $[a^\mu, b, a^\nu] = 0$.

In what follows we use (II) as a hypothesis; the further development will show that (III) is a consequence.

2. Before we undertake the actual investigation of A we point out that the mathematical foundation (equivalent to the physical basis, cf. footnote 1) may

[6] Cf. footnote 1.

also be based on $a \pm b$, λa, and a^μ instead of $a \pm b$, λa, and ab. For then we could define ab as $\frac{1}{2}((a+b)^2 - a^2 - b^2)$ from which it follows automatically that ab is commutative. Accordingly the postulation assumes the following form in this case:

In the algebra A with elements $a, b, c, \cdots$, one can form from the elements a and b the elements $a \pm b$, λa (λ a real number), and a^μ ($\mu = 1, 2, \cdots, a^1 = a$); all the usual rules of calculation hold for the operations $a \pm b$ and λa; and A has a finite linear basis. a^μ is a continuous function of a (that is, if we represent all the elements as linear aggregates of the basis elements, then the coefficients of a^μ depend continuously on those of a). A is "formally real", that is,

(I′) If $a^2 + b^2 + c^2 + \cdots = 0$, then $a = b = c = \cdots = 0$.

((I′) is the same as (I).) It is always the case that

(II′) $$(a^\mu + a^\nu)^2 = a^{2\mu} + a^{2\nu} + 2a^{\mu+\nu}.$$

((II′) is equivalent to (II), but it has the definition $a^1 = a$, $a^{\mu+1} = a\,a^\mu$ as a consequence.) Finally,

(*) $$(a+b+c)^2 + a^2 + b^2 + c^2 = (a+b)^2 + (a+c)^2 + (b+c)^2.$$

((*) is equivalent to $(a+b)c = ac + bc$, but because of the continuity of a^2 (and therefore also of ab), it has the condition $(\lambda a)b = \lambda(ab)$ as a consequence. This completes the list of the desired rules of calculation.—(*) could be replaced by the simpler rule $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$.

All of these postulates, except for (*), necessarily arise from the assumed physical conditions.

Part I. Structure of the r-number algebras.

3. We now begin the investigation of the properties of A.

THEOREM 1. *If $a \neq 0$, then $a^\mu \neq 0$.*

Proof: If $a^\nu \neq 0$, then, by (I), $(a^\nu)^2 \neq 0$. Therefore $a^{\nu+1} \neq 0$ since we have $a^{\nu+1}a^{\nu-1} = a^{2\nu} = (a^\nu)^2$. (If $\nu = 1$, the factor $a^{\nu-1}$ is to be omitted.) Hence if $a \neq 0$, then $a^\mu \neq 0$.

We call an element $e = e^2 \neq 0$ a unit element (or idempotent element). If two distinct unit elements e and e' are such that $ee' = 0$, we say that they are mutually orthogonal. If $ee' = 0$, then $e + e'$ is also a unit element (because $(e+e')^2 = e^2 + e'^2 + 2ee' = e + e'$ and $e(e+e') = e^2 + ee' = e \neq 0$ so that $e + e' \neq 0$).

THEOREM 2. *Corresponding to each $a \neq 0$ there exists a unit element $e = e(a)$ (where e is a polynomial in a, and $e = e^2 \neq 0$) such that, for $\mu \geqq 2$, $ea^\mu = a^\mu$.*

Proof: Since the basis of A is finite, only a finite number of the elements $a, a^2, \cdots$ are linearly independent. Suppose that a^μ depends linearly on $a, \cdots, a^{\mu-1}$ so that

$$a^\mu + \lambda_{\mu-1}a^{\mu-1} + \cdots + \lambda_\rho a^\rho = 0.$$

Since $a \neq 0$ and $a^\mu \neq 0$, it follows that $\lambda_\rho \neq 0$ and $\rho < \mu$ if ρ is given its largest possible value. The condition $\rho > 2$ would imply that

$$a^\mu + \lambda_{\mu-1}a^{\mu-1} + \cdots + \lambda_\rho a^\rho = (a^{\mu-2} + \lambda_{\mu-1}a^{\mu-3} + \cdots + \lambda_\rho a^{\rho-2})a^2 = 0,$$

$$(a^{\mu-2} + \lambda_{\mu-1}a^{\mu-3} + \cdots + \lambda_\rho a^{\rho-2})^2 a^2 = 0,$$

so that, by Theorem 1,

$$(a^{\mu-2} + \lambda_{\mu-1}a^{\mu-3} + \cdots + \lambda_\rho a^{\rho-2})a = a^{\mu-1} + \lambda_{\mu-1}a^{\mu-2} + \cdots + \lambda_\rho a^{\rho-1} = 0.$$

Therefore if μ is given its smallest possible value it follows that $\rho = 1$ or 2. Now put

$$e = e(a) = \frac{(a^{\mu-\rho} + \lambda_{\mu-1}a^{\mu-\rho-1} + \ldots + \lambda_{\rho+1}a)^\rho}{(-\lambda_\rho)^\rho}. \tag{5}$$

Then

$$\frac{a^{\mu-\rho} + \lambda_{\mu-1}a^{\mu-\rho-1} + \ldots + \lambda_{\rho+1}a}{-\lambda_\rho}\, a^\rho = a^\rho.$$

Hence $ea^\rho = a^\rho$, and $ea^{\mu'} = a^{\mu'}$ if $\mu' \geqq \rho$, and therefore *a fortiori* if $\mu' \geqq 2$. Hence $e^2 = e\,e(a) = e$, and since $ea^2 = a^2 \neq 0$, it follows that $e \neq 0$, i.e., e is a unit element. (Note that, having dealt in this proof only with powers and polynomials involving one element a, we could use the associative law because of (II).)

The simplest special case of (II) is $a^2a^2 = a^4$, that is, $a^2a^2 = (a^2a)a$. If we now replace a by $\lambda a + \mu b$ and compare the terms proportional to $\lambda^3\mu$ and $\lambda^2\mu^2$ respectively, then

$$4(ab)a^2 = a^3b + a(a^2b) + 2a(a(ab)), \tag{6}$$

$$4(ab)^2 + 2a^2b^2 = a(ab^2) + b(ba^2) + 2a(b(ab)) + 2b(a(ab)). \tag{7}$$

If in (6) we replace a by a unit element e and replace b by x, then

$$2e(e(ex)) - 3e(ex) + ex = 0. \tag{8}$$

THEOREM 3. *If e is a unit element and if $N_\lambda = N_\lambda(e)$ is the set of x's such that $ex = \lambda x$, then N_λ is obviously a linear manifold, but if $\lambda \neq 0, \frac{1}{2}$, and 1, then N_λ consists of zero alone.*

Each x can be represented in the form

$$x = x_0 + x_{\frac{1}{2}} + x_1, \qquad x_0 \in N_0, \qquad x_{\frac{1}{2}} \in N_{\frac{1}{2}}, \qquad x_1 \in N_1, \tag{9}$$

in one and only one way.

Proof: If $x \in N_\lambda$, then, by (8), $(2\lambda^3 - 3\lambda^2 + \lambda)x = 0$, and if $\lambda \neq 0, \frac{1}{2}$, and 1, then $x = 0$. If $x \in N_\lambda$, then $2e(ex) - 3ex + x = (2\lambda^2 - 3\lambda + 1)x$. For $\lambda = 0$ this expression reduces to x and for $\lambda = \frac{1}{2}$ or 1 it reduces to 0. Hence, by (9),

$$\begin{cases} x_0 = 2e(ex) - 3ex + x, \\ x_{\frac{1}{2}} = -4e(ex) + 4ex, \\ x_1 = 2e(ex) - ex, \end{cases} \tag{10}$$

where the latter two formulas are obtained in a similar way. Thus equation (9) can have no solution other than (10). But (10) is a solution, for the equations $ex_0 = 0$, $ex_{\frac{1}{2}} - \frac{1}{2}x_{\frac{1}{2}} = 0$, and $ex_1 - x_1 = 0$ are equivalent to (8), and it is evident that $x = x_0 + x_{\frac{1}{2}} + x_1$.

THEOREM 4. *If $a \in N_0$ and $b \in N_0$, then $ab \in N_0$; if $a \in N_1$ and $b \in N_1$, then $ab \in N_1$; if $a \in N_0$ and $b \in N_1$, then $ab = 0$. Thus N_0 and N_1 are mutually orthogonal sub-algebras of A. If $a \in N_{\frac{1}{2}}$ and $b \in N_{\frac{1}{2}}$, then $ab \in (N_0 + N_1)$; if $a \in N_{\frac{1}{2}}$ and $b \in N_0$, then[7] $ab \in (N_0 + N_{\frac{1}{2}})$; if $a \in N_{\frac{1}{2}}$ and $b \in N_1$, then[7] $ab \in (N_{\frac{1}{2}} + N_1)$.*

Proof: in (7) we set $a = e$ and replace b by $\lambda a + \mu b$. If we consider the terms proportional to $\lambda\mu$, then

$$8(ea)(eb) + 4e(db) - 2e(e(ab)) - (ea)b - (eb)a$$
$$- 2e(a(eb)) - 2e(b(ea)) - 2a(e(eb)) - 2b(e(ea)) = 0,$$

and if $a \in N_\rho$ and $b \in N_\sigma$, then

$$- 2e(e(ab)) + (4 - 2\rho - 2\sigma)e(ab) + (8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)ab = 0.$$

If we resolve ab according to Theorem 3, then

$$(8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)\,(ab)_0 + (8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \tfrac{3}{2})\,(ab)_{\frac{1}{2}}$$
$$+ (8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2)\,(ab)_1 = 0.$$

Since this may be regarded as a resolution of zero in the sense of Theorem 3, each of the three terms separately vanish. Hence if one of the three conditions

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma \neq 0, \qquad 8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \tfrac{3}{2} \neq 0,$$
$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2 \neq 0$$

is satisfied, the corresponding equation $(ab)_0 = 0$, $(ab)_{\frac{1}{2}} = 0$, $(ab)_1 = 0$ must be true. Substitution of $\rho, \sigma = 0, \frac{1}{2}$, and 1 leads to the desired results.

COROLLARY. *If $a \in N_{\frac{1}{2}}$, then, by Theorem 4, $a^2 \in (N_0 + N_1)$; yet were it the case that $a^2 \in N_0$ or $a^2 \in N_1$, then a would be zero.*

Proof: It would follow that $ea = \frac{1}{2}a$ and $ea^2 = \lambda a^2$ with $\lambda = 0$ or 1. Hence if $b = e$, then (6) becomes

$$2a^3 = ea^3 + \lambda a^3 + a^3,$$

so that $ea^3 = (1 - \lambda)a^3$ and $a^3 \in N_{1-\lambda}$. By Theorem 4, it follows that $a^5 = a^3a^2 = 0$ ($a^2 \in N_\lambda$, $a^3 \in N_{1-\lambda}$), so that $a = 0$.

THEOREM 5. *There exists one modulus 1, that is, one unit element $e = 1$ such that $1a = a$ for every a.*

Proof: We select a unit element e so that the resulting dimension δ of the algebra $N_0(e)$ is as small as possible. Suppose $N_0(e)$ contained an element $a \neq 0$. Then, by Theorem 2, it would contain a unit element e'. Since $e' \in N_0(e)$, it

[7] We shall later (Theorem 7) restrict this to $ab \in N_{\frac{1}{2}}$.

follows that $ee' = 0$. Hence $e + e'$ is also a unit element. Since $(e + e')e = e$, e belongs to $N_1(e + e')$, and if $a \in N_0(e + e')$ then, by Theorem 4, $ea = 0$ and $a \in N_0(e)$. Therefore $N_0(e + e') \subseteq N_0(e)$. $e' \in N_0(e)$, but e' does not belong to $N_0(e + e')$ since $(e + e')e' = e' \neq 0$. Therefore $N_0(e + e') \neq N_0(e)$. Hence the dimension of $N_0(e + e')$ is less than δ, contrary to assumption. Consequently $N_0(e)$ consists of 0 alone.

If $a \in N_1(e)$, then, by Theorem 4, $a^2 \in [N_0(e) + N_1(e)]$. By the above argument $a^2 \in N_1(e)$, and by the Corollary, $a = 0$. Hence $N_1(e)$ consists of 0 alone.

In the notation of (9) it follows that x always equals x_1, hence that $x \in N_1(e)$, and hence that $ex = x$. Therefore e is the modulus.

4. Because of the existence of the modulus we may restate in more complete form the considerations which led up to Theorem 2. Among the elements $1, a, a^2, \cdots$ linear dependence must occur so that there exists a relation of the form

$$f(a) = a^\mu + \lambda_{\mu-1}a^{\mu-1} + \cdots + \lambda_1 a + \lambda_0 1 = 0. \tag{11}$$

If this relation is such that μ has the smallest possible value, then we call (11) the *characteristic equation* and we call μ the *order* of a. These notions are uniquely determined, for if there existed two equations $f(a) = 0$ and $g(a) = 0$ of this form, then $f(a) - g(a) = 0$ would be an equation of lower degree unless the polynomials f and g were identical.

Let ξ be a variable whose values are real numbers. Suppose the equation $f(\xi) = 0$ had a complex root $\xi = \rho + i\sigma$ with $\sigma \neq 0$. Then it would also have the root $\xi = \rho - i\sigma$. Hence $f(\xi)$ would be divisible by the polynomial

$$(\xi - (\rho + i\sigma))(\xi - (\rho - i\sigma)) = (\xi - \rho)^2 + \sigma^2.$$

Again, if $f(\xi) = 0$ had a multiple real root $\xi = \rho$, then $f(\xi)$ would again be divisible by $(\xi - \rho)^2 + \sigma^2$, where $\sigma = 0$. In either case $f(\xi)$ is of the form

$$f(\xi) = ((\xi - \rho)^2 + \sigma^2)g(\xi).$$

If $f(a) = 0$, then $f(a)g(a) = 0$, and hence $((a - \rho 1)g(a))^2 + (\sigma g(a))^2 = 0$. It follows by (I) that $(a - \rho 1)g(a) = 0$. But this is impossible since $(\xi - \rho)g(\xi)$ is a polynomial of the same form as $f(\xi)$ but of lower degree.

It follows that $f(\xi) = 0$ has μ distinct simple real roots $\xi_1, \xi_2, \cdots, \xi_\mu$ which we call the *proper values* of a.

If two polynomials $g(\xi)$ and $h(\xi)$ have the same value at each of the points $\xi = \xi_1, \xi_2, \cdots, \xi_\mu$, then $g(\xi) - h(\xi)$ is divisible by $f(\xi) = (\xi - \xi_1) \cdots (\xi - \xi_\mu)$ so that $g(\xi) - h(\xi) = f(\xi)k(\xi)$. Hence

$$g(a) - h(a) = f(a)k(a) = 0 \quad \text{and} \quad g(a) = h(a).$$

THEOREM 6. *Each element a may be represented as a linear combination of a set of pairwise orthogonal unit elements whose sum is the modulus; the coefficients in this representation are the proper values of a.*

Proof: Let $p_\nu(\xi) = \dfrac{f(\xi)}{(\xi - \xi_\nu)f'(\xi_\nu)}$ $(\nu = 1, 2, \cdots, \mu)$ be that polynomial whose value is zero when ξ has any of the values $\xi_1, \xi_2, \cdots, \xi_\mu$ except ξ_ν, in which case $p_\nu(\xi)$ has the value unity. Since $p_\nu(\xi)$ is not identically zero and since the degree of $p_\nu(\xi)$ is less than μ, it follows that $p_\nu(a) \neq 0$. Let

$$e_\nu = p_\nu(a) \qquad (\nu = 1, 2, \cdots, \mu). \tag{12}$$

For each $\xi = \xi_1, \xi_2, \cdots, \xi_\mu$, $p_\nu(\xi)^2 = p_\nu(\xi)$ and $p_\nu(\xi)p_{\nu'}(\xi) = 0$, where $\nu \neq \nu'$; also $p_1(\xi) + \cdots + p_\mu(\xi) = 1$ and $\xi_1 p_1(\xi) + \cdots + \xi_\mu p_\mu(\xi) = \xi$. It follows from the last remark before Theorem 6 that

$$\begin{cases} e_\nu^2 = e_\nu \neq 0,\ e_\nu e_{\nu'} = 0 \text{ for } \nu \neq \nu', \\ e_1 + \cdots + e_\mu = 1, \\ \xi_1 e_1 + \cdots + \xi_\mu e_\mu = a. \end{cases} \tag{13}$$

This completes the proof of all parts of the theorem.

Since the relation $g(\xi_1)p_1(\xi) + \cdots + g(\xi_\mu)p_\mu(\xi) = g(\xi)$ holds for any polynomial $g(\xi)$ for each $\xi = \xi_1, \xi_2, \cdots, \xi_\mu$, then

$$g(\xi_1)e_1 + \cdots + g(\xi_\mu)e_\mu = g(a). \tag{14}$$

If $\xi_\nu \neq 0$, we can find a polynomial $h(\xi)$ which is zero for $\xi = 0, \xi_1, \xi_2, \cdots,$ and ξ_μ except ξ_ν, and which has the value unity for $\xi = \xi_\nu$. Then by (14), $h(a) = e_\nu$ and $h(\xi)$ contains no constant term. Hence $h(a)$ contains no term 1 but only terms $a, a^2, \cdots$. This proves the following

COROLLARY. *If a belongs to a sub-algebra B of A, then each of its e_ν whose $\xi_\nu \neq 0$ belongs to B.*

THEOREM 7. *Theorem 4 may be modified as follows: if $a \in N_{\frac{1}{2}}$ and if either $b \in N_0$ or $b \in N_1$, then $ab \in N_{\frac{1}{2}}$.*

Proof: By Theorem 6 and its corollary it is sufficient, since N_0 and N_1 are algebras, to consider only unit elements $b = e'$. $1 - e$ is a unit element, for $(1 - e)^2 = 1 - e$, and if $1 - e$ were zero, then it would be the case that $e = 1$ and $N_{\frac{1}{2}} = 0$, so that no proof would be necessary. Since the substitution of $(1 - e)$ for e carries N_0, $N_{\frac{1}{2}}$, and N_1 over into N_1, $N_{\frac{1}{2}}$, and N_0 respectively, it is sufficient to consider only those elements $b \in N_0$; for these elements, $ee' = 0$. (In this argument N_λ has really signified $N_\lambda(e)$.)

Since e and e' are orthogonal it follows that $e + e'$ is a unit element and according to (9) we are able to resolve a with respect to $e + e'$ as follows: $a = a_0 + a_{\frac{1}{2}} + a_1$. Since $a \in N_1(e)$ and $ea = \frac{1}{2}a$, we have

$$ea_0 + ea_{\frac{1}{2}} + ea_1 = \tfrac{1}{2}a_0 + \tfrac{1}{2}a_{\frac{1}{2}} + \tfrac{1}{2}a_1.$$

By application of Theorem 4 to $e + e'$ it follows that

$$ea_0 = 0, \qquad ea_{\frac{1}{2}} \in [N_0(e + e') + N_{\frac{1}{2}}(e + e')], \qquad ea_1 \in N_1(e + e')$$

since $(e + e')e = e$ and $e \epsilon N_1(e + e')$. From the right hand side of the above equation we have that

$$(\tfrac{1}{2}a_0 + \tfrac{1}{2}a_{\frac{1}{2}}) \epsilon [N_0(e + e') + N_{\frac{1}{2}}(e + e')], \qquad \tfrac{1}{2}a_1 \epsilon N_1(e + e').$$

Because of the uniqueness of the resolution (9) it results that

$$ea_{\frac{1}{2}} = \tfrac{1}{2}a_0 + \tfrac{1}{2}a_{\frac{1}{2}}, \qquad ea_1 = \tfrac{1}{2}a_1.$$

By definition,

$$(e + e')a_0 = 0, \qquad (e + e')a_{\frac{1}{2}} = \tfrac{1}{2}a_{\frac{1}{2}}, \qquad (e + e')a_1 = a_1,$$

so that

$$e'a_0 = 0, \qquad e'a_{\frac{1}{2}} = -\tfrac{1}{2}a_0, \qquad e'a_1 = \tfrac{1}{2}a_1.$$

Hence $e'(e'a_{\frac{1}{2}}) = 0$. Resolving $a_{\frac{1}{2}}$ by (9) with respect to e' shows that this implies $a_{\frac{1}{2}} \epsilon N_0(e')$. Therefore $-\tfrac{1}{2}a_0 = e'a_{\frac{1}{2}} = 0$ and $a_0 = 0$. Consequently

$$e'a = e'a_0 + e'a_{\frac{1}{2}} + e'a_1 = 0 + 0 + \tfrac{1}{2}a_1 = \tfrac{1}{2}a_1.$$

Since $ea_1 = \tfrac{1}{2}a_1$, $a_1 \epsilon N_1(e)$, and $e'a \epsilon N_{\frac{1}{2}}(e)$, we have finally that $ab \epsilon N_{\frac{1}{2}}(e)$.

Theorem 6 and Formula (14) give us the means to prove (III), that is, that $[a^2, b, a] = 0$. If we resolve a by Theorem 6 then, by (14), it is sufficient to prove the relation $[e_\nu, b, e_{\nu'}] = 0$ for all ν and ν'. If $\nu = \nu'$ this relation follows from (2), but if $\nu \neq \nu'$, then $e_\nu e_{\nu'} = 0$ and it remains to show that

$$e(e'b) = e'(eb) \tag{15}$$

if e and e' are orthogonal unit elements. By Theorem 3 it suffices to consider the cases $b \epsilon N_0(e)$, $b \epsilon N_{\frac{1}{2}}(e)$, and $b \epsilon N_1(e)$. $ee' = 0$ implies $e' \epsilon N_0(e)$, and so theorems 4 and 7 prove the statement.

5. In accordance with the usual general terminology, the algebra A is called *irreducible* if it contains no invariant subalgebra B, that is, if no subalgebra B (except A itself and 0) is such that if $a \epsilon A$ and $b \epsilon B$, then $ab \epsilon B$. A is called a *direct sum* if it is the sum of two orthogonal subalgebras B and C, that is, if every $a \epsilon A$ can be written in the form $a = b + c$, where $b \epsilon B$ and $c \epsilon C$ and where B and C are such that if $b \epsilon B$ and $c \epsilon C$, then $bc = 0$ (neither B nor C being A or 0).[8] Every r-number algebra is completely reducible, that is, the properties of being reducible and of being a direct sum are equivalent. For it is clear that B is an invariant subalgebra if A is the direct sum of B and C. Conversely, if B is an invariant subalgebra then it has a modulus 1^*, and from its invariance it follows immediately that $N_1(1^*) = B$, $N_{\frac{1}{2}}(1^*) \subset B$, so that $N_{\frac{1}{2}}(1^*) = 0$. Consequently A is the direct sum of B and C if $N_0(1^*) = C$.

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[8] If b belongs to B and C simultaneously, then $b^2 = 0$ and $b = 0$. Hence the resolution $a = b + c$ is unique, where $b \epsilon B$ and $c \epsilon C$.

We can represent every r-number algebra A as the direct sum of finitely many irreducible subalgebras $B_1, \cdots, B_k$ (all $\neq 0$), that is, each $a \in A$ is

$$a = b_1 + \cdots + b_k, \qquad b_1 \in B_1, \cdots, b_k \in B_k, \tag{16}$$

where if $b_i \in B_i$ and $b_j \in B_j$ for $i \neq j$, then $b_i b_j = 0$.[9] Moreover, $B_1, \cdots, B_k$ are uniquely determined except for their order. For let $C_1, \cdots, C_h$ be a second resolution of this kind. Since B_μ and C_ν are invariant subalgebras of A, so also is their intersection which is likewise an invariant subalgebra of B_μ and C_ν. Since both B_μ and C_ν are irreducible, their intersection must be 0 or else it must equal B_μ and C_ν. Hence if $B_\mu = C_\nu$, the intersection of B_μ and C_ν is zero. The sum of all the C_ν is A which in turn contains B_μ. Thus not all of these intersections can be zero. Hence for each μ there is a ν such that $B_\mu = C_\nu$. Because the C_ν are pairwise orthogonal there is only one such ν. Similarly for each ν there is one and only one μ such that $B_\mu = C_\nu$. Hence $k = h$ and the B_μ go over by a permutation into the C_ν.

The $B_1, \cdots, B_k$ are the *irreducible components* of A. It is apparent from the above discussion that it is sufficient to investigate the irreducible algebras A since the other algebras may be constructed from these in the simple manner just indicated. Consequently we shall confine our interest merely to the irreducible algebras, although we shall word our theorems for all algebras wherever the proofs do not make use of irreducibility.

6. We now examine systems of pairwise orthogonal unit elements $e_1, \cdots, e_\gamma$ ($e_\rho^2 = e_\rho \neq 0$, $e_\rho e_\sigma = 0$ for $\rho \neq \sigma$).

THEOREM 8. *Let $e_1, \cdots, e_\gamma$ be a system of pairwise orthogonal unit elements with the sum 1. Let the set of all x such that*[10]

$$e_\tau x = \tfrac{1}{2}(\delta_{\rho\tau} + \delta_{\sigma\tau})x \qquad (\tau = 1, \cdots, \gamma) \tag{17}$$

be denoted by $M^{\rho\sigma}$, where $\rho, \sigma = 1, \cdots, \gamma$.[11] *$M^{\rho\sigma}$ is obviously a linear manifold.*

Every x can be represented in the form

$$x = \sum_{\rho \leqq \sigma} x_{\rho\sigma}, \qquad x_{\rho\sigma} \in M^{\rho\sigma}, \tag{18}$$

in one and only one way.

Proof: It follows from (17) and (18) that

$$\begin{cases} 4e_\rho(e_\sigma x) = 4e_\sigma(e_\rho x) = x_{\rho\sigma} & (\rho \neq \sigma), \\ 2e_\rho(e_\rho x) - e_\rho x = x_{\rho\rho}. \end{cases} \tag{19}$$

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[9] Consequently the resolution (16) is unique. Cf. footnote 8.

[10] $\delta_{\mu\nu}$ is the Kronecker symbol, that is, $\delta_{\mu\nu} = \begin{cases} 1 \text{ for } \mu = \nu, \\ 0 \text{ for } \mu \neq \nu. \end{cases}$

[11] It is obvious that $M^{\rho\sigma} = M^{\sigma\rho}$, so that for the most part we shall be able to limit ourselves to the consideration of $M^{\rho\sigma}$ for $\rho \leqq \sigma$.

Therefore (18) can have no solution other than (19) and, as we shall show, (19) is a solution of (18). ((15) shows that (19) is self-consistent.) Comparison of (19) with (10) shows that $x_{\rho\rho}$ belongs to $N_1(e_\rho)$. For $\tau \neq \rho$, $e_\tau e_\rho = 0$ so that these e_τ belong to $N_0(e_\rho)$. Hence for $\tau \neq \rho$, $e_\rho x_{\rho\rho} = x_{\rho\rho}$ and $e_\tau x_{\rho\rho} = 0$. Hence by (17), $x_{\rho\rho} \in M^{\rho\rho}$. If in (4) we set $a = b = e_\rho$, $c = x$, and $u = e_\sigma$, then for $\rho \neq \sigma$ (since $e_\rho^2 = e_\rho$ and $e_\rho e_\sigma = 0$)

$$[e_\rho, e_\sigma, x] + 2[e_\rho x, e_\sigma, e_\rho] = 0,$$

$$-e_\rho(e_\sigma x) + 2e_\rho(e_\sigma(e_\rho x)) = 0,$$

$$e_\rho(e_\sigma(e_\rho x)) = \tfrac{1}{2}e_\rho(e_\sigma x), \qquad e_\rho x_{\rho\sigma} = \tfrac{1}{2}x_{\rho\sigma}.$$

By symmetry it follows that $e_\sigma x_{\rho\sigma} = \frac{1}{2}x_{\rho\sigma}$. Hence $(e_\rho + e_\sigma)x_{\rho\sigma} = x_{\rho\sigma}$ and $x_{\rho\sigma} \in N_1(e_\rho + e_\sigma)$. If $\tau \neq \rho$ and σ, then $(e_\rho + e_\sigma)e_\tau = 0$ and $e_\tau \in N_0(e_\rho + e_\sigma)$ so that $e_\tau x_{\rho\sigma} = 0$. By (17), $x_{\rho\sigma} \in M^{\rho\sigma}$. Finally, $\left(\text{since } \sum_\rho e_\rho = 1\right)$

$$\sum_{\rho \leqq \sigma} x_{\rho\sigma} = \sum_\rho (2e_\rho(e_\rho x) - e_\rho x) + \sum_{\rho < \sigma} (2e_\rho(e_\sigma x) + 2e_\sigma(e_\rho x))$$

$$= 2\sum_{\rho,\sigma} e_\rho(e_\sigma x) - \sum_\rho e_\rho x = 2x - x = x.$$

This completes the proof of all parts of the theorem.

THEOREM 9. *Suppose $a \in M^{\rho\sigma}$ and $b \in M^{\pi\tau}$. Then: if both ρ and σ are distinct from both π and τ, then $ab = 0$; if ρ, σ and π, τ have one common index, say $\rho = \pi$ but $\sigma \neq \tau$, then $ab \in M^{\sigma\tau}$; if ρ, σ are the same pair as π, τ (except possibly for order; cf. footnote 11), say $\rho = \pi$ and $\sigma = \tau$, then $ab \in (M^{\rho\rho} + M^{\sigma\sigma})$ for $\rho \neq \sigma$ and $ab \in M^{\rho\rho}$ for $\rho = \sigma$.*

Proof: We set

$$e = \begin{cases} e_\rho + e_\sigma \text{ for } \rho \neq \sigma, \\ e_\rho \qquad \text{for } \rho = \sigma, \end{cases} \qquad e' = \begin{cases} e_\pi + e_\tau \text{ for } \pi \neq \tau, \\ e_\pi \qquad \text{for } \pi = \tau. \end{cases}$$

By (17) it follows that $ea = a$ and $e'b = b$ so that $a \in N_1(e)$ and $b \in N_1(e')$. If both ρ and σ are distinct from both π and τ, then it is obvious that $ee' = 0$ and $e' \in N_0(e)$. Hence, by Theorem 4, $e'a = 0$, $a \in N_0(e')$, and $ab = 0$.

Suppose that σ, τ, and $\rho = \pi$ are three distinct numbers. By Theorem 4, $B = N_1(e_\rho + e_\sigma + e_\tau)$ is an algebra and its modulus is $e_\rho + e_\sigma + e_\tau$. If we resolve the condition $(e_\rho + e_\sigma + e_\tau)x = x$ according to (17) and (18), the result shows that x can contain only those addends $x_{\mu\nu}$ in which $\mu, \nu = \rho, \sigma, \tau$. Hence $B = N_1(e_\rho + e_\sigma + e_\tau) = \sum_{\mu,\nu = \rho,\sigma,\tau} M^{\mu\nu}$. In a similar way it is apparent that the relations $N_1(e_\rho + e_\sigma) = \sum_{\mu,\nu = \rho,\sigma} M^{\mu\nu}$, $N_{\frac{1}{2}}(e_\rho + e_\sigma) = \sum_{\mu = \rho,\sigma} M^{\mu\tau}$, and $N_0(e_\rho + e_\sigma) = M^{\tau\tau}$ hold in B. So $a \in N_1$ and $b \in N_{\frac{1}{2}}$ and, by Theorem 7, $ab \in N_{\frac{1}{2}}$, where $N_{\frac{1}{2}} = M^{\rho\tau} + M^{\sigma\tau}$. By symmetry it follows that $ab \in (M^{\rho\sigma} + M^{\sigma\tau})$. Because of the uniqueness of the resolution (18), $ab \in M^{\sigma\tau}$.

If $\rho = \pi = \sigma \neq \tau$, we reach the same result by considering $B = N_1(e_\rho + e_\tau)$ and $N_1(e_\rho)$, $N_{\frac{1}{2}}(e_\rho)$, and $N_0(e_\rho)$. By symmetry the same result holds for $\rho = \pi = \tau \neq \sigma$. This accounts for all the cases where $\rho = \pi$ and $\sigma \neq \tau$.

If $\rho = \pi$ and $\sigma = \tau$, then $e = e'$. It follows that $a \in N_1(e)$ and $b \in N_1(e)$ so that $ab \in N_1(e)$. If $\rho = \sigma$, then $ab \in N_1(e_\rho)$, where $N_1(e_\rho) = M^{\rho\rho}$. If $\rho \neq \sigma$, then $ab \in N_1(e_\rho + e_\sigma)$, where $N_1(e_\rho + e_\sigma) = \sum_{\mu,\nu = \rho,\sigma} M^{\mu\nu} = M^{\rho\rho} + M^{\sigma\sigma} + M^{\rho\sigma}$. The relations $N_1(e_\rho) = M^{\rho\rho}$, $N_{\frac{1}{2}}(e_\rho) = M^{\rho\sigma}$, and $N_0(e_\rho) = M^{\sigma\sigma}$ (note (17) and (18)) hold in $B = N_1(e_\rho + e_\sigma)$. Hence $a \in N_{\frac{1}{2}}$, $b \in N_{\frac{1}{2}}$, and, by Theorem 4, $ab \in (N_0 + N_1)$, where $N_0 + N_1 = M^{\rho\rho} + M^{\sigma\sigma}$. This completes the proof of all parts of the theorem.

A unit element e is called *unresolvable* if it can not be represented as the sum of two orthogonal unit elements.

THEOREM 10. *Every unit element can be represented as the sum of pairwise orthogonal unresolvable unit elements.*

Proof: There exist representations of e as a sum of pairwise orthogonal unit elements:

$$e_1 + \cdots + e_\gamma = e.$$

For example, $e_1 = e$, $\gamma = 1$. Since $e_1, \cdots, e_\gamma$ are linearly independent (if $\lambda_1 e_1 + \cdots + \lambda_\gamma e_\gamma = 0$, then $e_\mu(\lambda_1 e_1 + \cdots + \lambda_\gamma e_\gamma) = \lambda_\mu e_\mu = 0$ and $\lambda_\mu = 0$ for all $\mu = 1, \cdots, \gamma$), then $\gamma \leqq N$, where N is the dimension of A. Let γ be as large as possible. Suppose an e_μ ($\mu = 1, \cdots, \gamma$) were resolvable, $e_\mu = e'_\mu + e''_\mu$. Then we would have

$$e_1 + \cdots + e'_\mu + e''_\mu + \cdots + e_\gamma = e.$$

Since $e_\mu e'_\mu = (e'_\mu + e''_\mu)e'_\mu = e'_\mu$, therefore $e'_\mu \in N_1(e_\mu)$. Likewise $e''_\mu \in N_1(e_\mu)$. For $\mu \neq \nu$, $e_\mu e_\nu = 0$ and $e_\nu \in N_0(e_\mu)$ so that $e'_\mu e_\nu = e''_\mu e_\nu = 0$. Thus the $\gamma + 1$ elements $e_\mu, \cdots, e'_\mu, e''_\mu, \cdots, e_\gamma$ are pairwise orthogonal. This contradicts our original assumption and every e_μ must be unresolvable.

THEOREM 11. *A unit element e is unresolvable when and only when the dimension of $N_1(e)$ is equal to unity.*

Proof: If e is resolvable, $e = e' + e''$, then $ee' = (e' + e'')e' = e'$ and $e' \in N_1(e)$. Likewise $e'' \in N_1(e)$, and e' and e'' are linearly independent. Hence the dimension is greater than unity.

Conversely, suppose that e is unresolvable. If $a \in N_1(e)$ then, in the resolution of a according to (13), each e_ν (for which $\xi_\nu \neq 0$) belongs to $N_1(e)$ by the corollary of Theorem 6. Hence $ee_\nu = e_\nu$. Since $e_\nu + (e - e_\nu) = e$, $(e - e_\nu)^2 = e - e_\nu$, and $e_\nu(e - e_\nu) = 0$, it follows that if $(e - e_\nu)$ were different from zero then e would be resolvable. Hence $e_\nu = e$. Hence there is at most one $\xi_\nu \neq 0$ so that $a = 0$ or $a = \xi_\nu e$. Therefore $N_1(e)$ contains only the elements λe and the dimension is unity.

On the basis of Theorem 10 we now assume a resolution of 1 in pairwise orthogonal unresolvable unit elements:

$$e_1 + \cdots + e_\gamma = 1,$$

and we form the corresponding $M^{\rho\sigma}$ in accordance with Theorem 8. Since $N_1(e_\rho) = M^{\rho\rho}$, then, by Theorem 11, each $M^{\rho\rho}$ has the dimension unity.

THEOREM 12. *Under the above assumptions* $a^2 = \alpha(e_\rho + e_\sigma)$, *where* a *runs through* $M^{\rho\sigma}$ *and* $\rho \neq \sigma$.

Proof: By Theorem 9 it follows that $a^2 \epsilon (M^{\rho\rho} + M^{\sigma\sigma})$ and, by Theorem 11, $M^{\rho\rho}$ consists merely of the elements αe_ρ. Likewise $M^{\sigma\sigma}$ consists of the elements βe_σ, so that $a^2 = \alpha e_\rho + \beta e_\sigma$. By (III) and (17),

$$(a^2 e_\rho)a = a^2(e_\rho a), \qquad \alpha e_\rho \cdot a = (\alpha e_\rho + \beta e_\sigma)\cdot(e_\rho a),$$

$$\alpha \cdot \tfrac{1}{2}a = (\alpha + \beta)\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}a, \qquad \tfrac{1}{4}(\alpha - \beta)\cdot a = 0,$$

so that if $a \neq 0$ then $\frac{1}{4}(\alpha - \beta) = 0$ and $\alpha = \beta$. If $a = 0$ then it follows immediately that $\alpha = \beta = 0$.

THEOREM 13. *Under the above assumptions there exists a linear basis* $s_1, \cdots, s_\chi$ (*or if it is desired to show the dependency of the basis on* ρ *and* σ, $s_1^{\rho\sigma}, \cdots, s_{\chi_{\rho\sigma}}^{\rho\sigma}$) *of* $M^{\rho\sigma}$ *such that the following multiplication rules are valid* (*see footnote* 10):

$$s_\mu s_\nu = \delta_{\mu\nu}(e_\mu + e_\nu) \qquad (\mu, \nu = 1, \cdots, \chi). \tag{20}$$

This basis is uniquely determined to within a (*real*) *orthogonal transformation*

$$s'_\mu = \sum_{\nu=1}^{\chi} \omega_{\mu\nu} s_\nu \qquad (\mu = 1, \cdots, \chi), \tag{21}$$

where the matrix $\|\omega_{\mu\nu}\|$ *is orthogonal.*

Proof: Since $ab = \frac{1}{2}((a + b)^2 - a^2 - b^2)$ it follows from Theorem 12 that ab likewise has the form $\alpha(e_\rho + e_\sigma)$. Therefore $\alpha = \alpha(a, b)$ is obviously linear in a and in b, is symmetric in a and b, and is consequently a symmetric bilinear form in a and b (that is, in the coefficients appearing in the expansion of ab with respect to $s_1, \cdots, s_\chi$). If an $\alpha(a, a)$ were not greater than zero, then it would be the case that

$$a^2 + (\sqrt{-\alpha(a, a)}\ (e_\rho + e_\sigma))^2 = \alpha(a, a)\ (e_\rho + e_\sigma) - \alpha(a, a)\ (e_\rho + e_\sigma) = 0$$

so that, by (I), $a = 0$. Hence if $a \neq 0$ then $\alpha(a, a) > 0$. Therefore the quadratic form $\alpha(a, a)$ is positive definite. Consequently $\alpha(a, b)$ can be transformed into the unit form by a linear transformation. Hence it is possible to select the basis of $M^{\rho\sigma}$ so that $\alpha(a, b)$ is the unit form of the expansion coefficients of a and b. $\alpha(s_\mu, s_\nu) = \delta_{\mu\nu}$ and (20) is proved.

It is clear that (20) is invariant under (21).

Formulas (17) and (20) together with the first and second multiplication rules of Theorem 9 (where we interchange ρ and σ) lead to the following multiplication table, providing that e_ρ and $s_\mu^{\rho\sigma}$ are selected in accordance with Theorems 12 and 13:

THEOREM 14. *Under the above assumptions it follows that* e_ρ $(\rho = 1, \cdots, \gamma)$ *and* $s_\mu^{\rho\sigma}$ $(\rho, \sigma = 1, \cdots, \gamma; \mu = 1, \cdots, \chi_{\rho\sigma}$, *where* $\rho \neq \sigma$ *and the pairs* ρ, σ *and* σ, ρ *are to be identified; see footnote* 11) *is a linear basis of* A *for which the following multiplication rules hold*:

$$
\begin{cases}
\text{(a)} & e_\rho\, e_\sigma = \begin{cases} e_\rho \text{ for } \rho = \sigma, \\ 0 \text{ for } \rho \neq \sigma, \end{cases} \\
\text{(b)} & e_\rho\, s_\mu^{\sigma\tau} = \begin{cases} \frac{1}{2}\, s_\mu^{\sigma\tau} \text{ for } \rho = \sigma \text{ or } \tau, \\ 0 \text{ for } \rho \neq \sigma \text{ and } \tau, \end{cases} \\
\text{(c)} & s_\mu^{\rho\sigma}\, s_\nu^{\rho\sigma} = \begin{cases} e_\rho + e_\sigma \text{ for } \mu = \nu, \\ 0 \text{ for } \mu \neq \nu, \end{cases} \\
\text{(d)} & s_\mu^{\rho\sigma}\, s_\nu^{\sigma\tau} = \sum_{\lambda=1}^{\chi_{\rho\tau}} A_{\nu\lambda\mu}^{\rho\sigma\tau}\, s_\lambda^{\rho\tau} \text{ for } \rho \neq \tau, \\
\text{(e)} & s_\mu^{\rho\sigma}\, s_\nu^{\tau\upsilon} = 0 \text{ for } \rho, \sigma \neq \tau, \upsilon.
\end{cases} \tag{22}
$$

The proof was given above.

7. The algebra A is completely described by (22) so that the only data we need are the integers γ and $\chi_{\rho\sigma}$ and the real numbers $A_{\nu\lambda\mu}^{\rho\sigma\tau}$. Our next problem is to deduce from the r-number character of A some conditions which will limit these numbers more closely.

THEOREM 15. *If ρ, σ, τ, and υ are four distinct numbers then, under the above assumptions,*

$$A_{\nu\lambda\mu}^{\rho\sigma\tau} = A_{\lambda\mu\nu}^{\sigma\tau\rho} = A_{\mu\nu\lambda}^{\tau\rho\sigma} = A_{\lambda\nu\mu}^{\sigma\rho\tau} = A_{\nu\mu\lambda}^{\rho\tau\sigma} = A_{\mu\lambda\nu}^{\tau\sigma\rho}, \tag{23}$$

$$\sum_{\nu=1}^{\chi_{\sigma\tau}} (A_{\nu\lambda\mu}^{\rho\sigma\tau}\, A_{\nu\lambda'\mu'}^{\rho\sigma\tau} + A_{\nu\lambda\mu'}^{\rho\sigma\tau}\, A_{\nu\lambda'\mu}^{\rho\sigma\tau}) = \tfrac{1}{2}\, \delta_{\lambda\lambda'}\, \delta_{\mu\mu'} \quad \text{(see footnote 10)}, \tag{24}$$

$$\sum_{\mu=1}^{\chi_{\sigma\upsilon}} A_{\mu\theta_\iota}^{\rho\sigma\upsilon}\, A_{\kappa\mu\lambda}^{\sigma\tau\upsilon} = \sum_{\nu=1}^{\chi_{\rho\tau}} A_{\kappa\theta\nu}^{\rho\tau\upsilon}\, A_{\lambda\nu\iota}^{\rho\sigma\tau}. \tag{25}$$

Proof: We substitute for a, b, c, and u in (4) the following combinations of the elements of the basis in Theorem 14: $s_\mu^{\rho\sigma}, s_\nu^{\sigma\tau}, e_\tau, s_\lambda^{\rho\tau}$; $s_\mu^{\rho\sigma}, s_{\mu'}^{\rho\sigma}, s_\lambda^{\rho\tau}, e_\tau$; $s_\iota^{\rho\sigma}, s_\lambda^{\sigma\tau}, s_\kappa^{\tau\upsilon}, e_\rho$. The first substitution gives $A_{\nu\lambda\mu}^{\rho\sigma\tau} = A_{\mu\nu\lambda}^{\tau\rho\sigma}$ and (23) follows by cyclic permutation of the indices. (24) follows from the second substitution with the aid of (23). The third substitution leads to (25).

Although the relations (23), (24), and (25) are merely necessary and not sufficient, yet they will enable us in what follows, particularly in Part III, to determine completely all r-number algebras.

We now draw a few conclusions.

THEOREM 16. *If $\chi_{\rho\tau} \neq 0$, then under the above assumptions $\chi_{\rho\sigma} = \chi_{\rho\tau}$.*

Proof: Since $\chi_{\rho\tau} \geqq 1$, we can set $\lambda = \lambda' = 1$ in (24) and it follows that $2\sum_{\nu=1}^{\chi_{\sigma\tau}} A_{\nu 1\mu}^{\rho\sigma\tau}\, A_{\nu 1\mu'}^{\rho\sigma\tau} = \frac{1}{2}\delta_{\mu\mu'}$. Therefore the vectors $2A_{11\mu}^{\rho\sigma\tau}, \cdots, 2A_{\chi_{\sigma\tau} 1\mu}^{\rho\sigma\tau}$, where $\mu = 1, \cdots, \chi_{\rho\sigma}$, form a normalized orthogonal system. Since their dimension

is $\chi_{\sigma\tau}$ and since there are $\chi_{\rho\sigma}$ of these vectors, $\chi_{\rho\sigma} \leqq \chi_{\sigma\tau}$. Interchange of ρ and τ gives $\chi_{\sigma\tau} \leqq \chi_{\rho\sigma}$, whence $\chi_{\rho\sigma} = \chi_{\sigma\tau}$.

THEOREM 17. *If A is irreducible then, under the above assumptions, all the $\chi_{\rho\sigma}$ are equal to each other and distinct from zero.*

Proof: Let N be the set of all ρ for which a chain $\rho_0 = 1, \rho_1, \cdots, \rho_{\epsilon-1}, \rho_\epsilon = \rho$ $(\epsilon = 0, 1, 2, \cdots)$ exists such that every $\chi_{\rho_{\eta-1}\rho_\eta} \neq 0$ for $\eta = 1, \cdots, \epsilon$. Let M be the set of all other ρ. N is not empty since $1 \in N$. If $\rho \in N$ and $\sigma \in M$, then $\chi_{\rho\sigma} = 0$. Hence $M^{\rho\sigma}$ contains only the element 0. $e = \sum_{\rho \in N} e_\rho$ is a unit element. By (17) and (18) it follows that the condition $ex = \frac{1}{2}x$ is equivalent to the condition that, in the resolution (18) of x, only those addends $x_{\rho\sigma}$ occur for which $\rho \in N$ and $\sigma \in M$. Hence $x_{\rho\sigma} \in M^{\rho\sigma}$ and $x_{\rho\sigma} = 0$ so that $x = 0$. Therefore $N_{\frac{1}{2}}(e) = 0$ and A is the sum of the two orthogonal subalgebras $N_0(e)$ and $N_1(e)$. Since A is irreducible and $N_1(e) \neq 0$, then $N_0(e) = 0$. But every e_ρ, $\rho \in M$, is distinct from zero and belongs to $N_0(e)$ (since $ee_\rho = 0$). Hence M is empty and every ρ, $\rho = 1, \cdots, \gamma$, belongs to N. Thus we can set up the chain $\rho_0 = 1$, $\rho_1, \cdots, \rho_{\epsilon-1}, \rho_\epsilon = \rho$ for every ρ.

From Theorem 16 it follows that $\chi_{\rho_{\eta-1}\sigma} = \chi_{\rho_\eta\sigma}$ for every σ since $\chi_{\rho_{\eta-1}\rho_\eta} \neq 0$. Therefore $\chi_{\rho_0\sigma} = \chi_{\rho_\epsilon\sigma}$ and $\chi_{\rho\sigma} = \chi_{1\sigma}$. But we can write $\chi_{1\sigma}$ as $\chi_{\sigma 1}$. By the preceding argument it follows that $\chi_{1\sigma} = \chi_{11}$ so that $\chi_{\rho\sigma} = \chi_{11}$.

If we denote the common value of the $\chi_{\rho\sigma}$ by χ, then Theorem 14 provides the following condition on the dimension N of A:

$$N = \gamma + \frac{\gamma(\gamma - 1)}{2}\chi, \qquad \chi > 0. \tag{26}$$

It should be noted that, while N is uniquely determined by A, this by no means needs to be the case for γ and χ as introduced in Theorems 12, 13, 14 and 17. We shall see later (Theorem 23 in Part II) that γ and χ are actually so determined.

Part II. Trace, Commutativity

8. The investigation of the concepts mentioned in the title of this part is not absolutely necessary for the logical connection of the entire paper since in Part III we shall determine directly from Part I all r-number algebras. And in the concrete instances taken up in Part III these concepts can be set up directly without reference to the general theory. But it is our desire to develop in general form for the sake of eventual generalizations our present concepts which are in themselves not uninteresting. Moreover we arrive at the proof of the determinateness of γ and χ. Finally we touch on the application of these concepts to the forming of quantum mechanical concepts.

All the following considerations arise from the setting up of the operation $O_a x = ax$, where $x \in A$. If A has a linear basis $a_1, \cdots, a_N$ then a matrix $\|T_{\alpha\beta}(a)\|$, $\alpha, \beta = 1, \cdots, N$, corresponds to O_a:

$$a\,a_\alpha = \sum_{\beta=1}^{N} T_{\alpha\beta}(a)\,a_\beta \qquad (\beta = 1, \ldots, N). \tag{27}$$

If the basis is altered then $\|T_{\alpha\beta}(a)\|$ is merely transformed as follows:

$$\begin{cases} a'_\alpha = \sum_{\beta=1}^{N} \psi_{\alpha\beta}\,a_\beta, & \beta = 1, \ldots, N, \\ \|T'_{\alpha\beta}(a)\| = \|\psi_{\alpha\beta}\| \cdot \|T_{\alpha\beta}(a)\| \cdot \|\psi_{\alpha\beta}\|^{-1}. \end{cases} \tag{28}$$

Therefore the concepts which are invariant under transformations, such as trace $\|T_{\alpha\beta}(a)\| = \sum_{\alpha=1}^{N} T_{\alpha\alpha}(a)$ and the commutativity of $\|T_{\alpha\beta}(a)\|$ and $\|T_{\alpha\beta}(b)\|$ (that is, of O_a and O_b), are independent of the selection of the basis.

9. Because of the non-associativity of multiplication in A it need not be the case that O_aO_b is either O_{ab} or O_{ba}, for otherwise it would be necessary that the relation $a(bx) = (ab)x$, that is, $[a, b, x] = 0$, hold for all x. From $ab = ba$ it does not follow that $O_aO_b = O_bO_a$ (that is, the commutativity of the corresponding matrices), for this would imply the relations $a(bx) = b(ax)$ and $a(xb) = (ax)b$, that is, $[a, x, b] = 0$ for all x. Consequently it is impossible to define even the general commutativity of multiplication as a concept of interchangeability in A.

Accordingly we call a and b *interchangeable* if the operations O_a and O_b are interchangeable, that is, if the matrices $\|T_{\alpha\beta}(a)\|$ and $\|T_{\alpha\beta}(b)\|$ are interchangeable, and hence if

$$[a, x, b] = 0 \tag{29}$$

holds for all x. The *centrum* of A consists of those elements a with which all elements b are interchangeable and hence which are such that

$$[a, x, y] = 0 \tag{30}$$

holds for all x and y. In this event it follows by (2) and (3) that

$$[x, a, y] = 0 \text{ and } [x, y, a] = 0. \tag{31}$$

With regard to the preceding definition of interchangeability it must be kept in mind that it is essential that the basis of A be finite. In important instances of quantum mechanics analogous to the r-number algebras the situation is quite otherwise. There the elements $a, b, c, \cdots$ are infinite Hermitian matrices $A, B, C, \cdots$ and our product

$$AB = \frac{A \times B + B \times A}{2},$$

where $A \times B$ is the usual associative non-commutative matrix multiplication. (See footnote 1 as well as the discussion in the introduction.) A simple calculation shows that

$$[A, B, C] = \frac{B \times (A \times C - C \times A) - (A \times C - C \times A) \times B}{4},$$

so that interchangeability occurs in our sense only when $A \times C - C \times A = \lambda 1$. It follows from this that $A \times C - C \times A = 0$ only for finite matrices but not for infinite matrices.[12] In generalizations to infinite r-number algebras it is necessary to exercise caution with regard to interchangeability. It is clear that the concept of centrum must remain unaltered.

THEOREM 18. *The elements b which are interchangeable with a form an algebra $C(a)$ which contains* 1. *If one represents a according to Theorem* 6 *and formula* (13), *then the algebras $N_1(e_1), \cdots, N_1(e_\mu)$ are pairwise orthogonal and $C(a)$ is their (direct) sum.*

Proof: For the resolution (18) of a with respect to the elements $e_1, \cdots, e_\mu$ it follows that $N_1(e_\nu) = M^{\nu\nu}$ and $N_1(e_{\nu'}) = M^{\nu'\nu'}$. By Theorem 9 these two algebras (cf. Theorem 4) are orthogonal for $\nu \neq \nu'$. Consequently $N_1(e_1) + \cdots + N_1(e_\mu)$ is an algebra which contains $1 = e_1 + \cdots + e_\mu$.

To show that a is interchangeable with each element of $N_1(e_1) + \cdots + N_1(e_\mu)$ it is sufficient to show that it is interchangeable with each element of every $N_1(e_{\nu'})$. Since $a = \xi_1 e_1 + \cdots + \xi_\mu e_\mu$ it is sufficient to show that each $x \in N_1(e_{\nu'})$ is interchangeable with each e_ν. By the above argument it follows that $x \in N_1(e_\nu)$ for $\nu = \nu'$ and that $x \in N_0(e_\nu)$ for $\nu \neq \nu'$. We now have to show that $[x, y, e_\nu] = 0$ for all y, but by Theorem 3 it suffices to consider the cases $y \in N_0(e_\nu)$, $y \in N_{\frac{1}{2}}(e_\nu)$, and $y \in N_1(e_\nu)$. In other words it remains to show that $(xy)e = x(ye)$ for all six combinations of $x \in N_0(e)$ and $x \in N_1(e)$ with $y \in N_0(e)$, $y \in N_{\frac{1}{2}}(e)$, and $y \in N_1(e)$. But this follows immediately from Theorems 4 and 7.

Conversely, suppose that b is interchangeable with a. We set $x = e_{\nu'}$ in $[a, x, b] = 0$ and resolve b according to (18) with respect to the $e_1, \cdots, e_\mu$ corresponding to a. By (17) one concludes immediately that $[e_\nu, e_{\nu'}, b^{\rho\sigma}]$, i.e., $(e_\nu e_{\nu'})b^{\rho\sigma} - e_\nu(e_{\nu'}b^{\rho\sigma})$ differs from zero only if $\rho \neq \sigma$, $\nu = \rho$, and $\nu' = \sigma$ (or, what amounts to the same thing, $\nu = \sigma$ and $\nu' = \rho$) in which case it equals $-\frac{1}{4}b^{\rho\sigma}$. Consequently

$$[a, e_{\nu'}, b] = -\frac{1}{4}\sum_{\nu \neq \nu'} \xi_\nu b^{\nu\nu'}.$$

Therefore $b^{\nu\nu'} = 0$ for $\nu \neq \nu'$ if ξ_ν is not zero. So $b^{\nu\nu'} \neq 0$, $\nu \neq \nu'$ implies $\xi_\nu = 0$. By symmetry, it implies $\xi_{\nu'} = 0$. But this contradicts the fact that all the ξ_ν are distinct (note the discussion preceding Theorem 6). Hence

$$b = \sum_{\nu=1}^{\mu} b^{\nu\nu} \text{ and } b \in (N_1(e_1) + \ldots + N_1(e_\mu)).$$

---

[12] For example, A and C may be the matrices arising from the well-known quantum mechanical operators x and $i\frac{d}{dx}$.

THEOREM 19. *The centrum Z is an algebra which contains* 1. *Its degree is the number k of the irreducible components of A.* (*Note the end of paragraph* 5.)

Proof: By Theorem 18, the first assertion follows from the fact that Z is the common part of all $C(a)$. It can also be deduced directly from the identity[13]

$$[ab, x, y] = [a, bx, y] - [a, b, xy] + a[b, x, y] + [a, b, x]y.$$

If $A = B_1 + \cdots + B_k$ is the representation of A as the direct sum of irreducible algebras, then let $\bar{e}_1, \cdots, \bar{e}_k$ be the moduli of $B_1, \cdots, B_k$. $\bar{e}_i^2 = \bar{e}_i \neq 0$ and $\bar{e}_i\bar{e}_j = 0$ for $i \neq j$ (B_i and B_j orthogonal); if $x_j \in B_j$, then $\bar{e}_i x_j = x_j$ for $i = j$ and $\bar{e}_i x_j = 0$ for $i \neq j$ (also because of orthogonality). Thus $(\bar{e}_1 + \cdots \bar{e}_k)\, x = x$ for $x = x_j$ and accordingly for every x. Hence $\bar{e}_1 + \cdots + \bar{e}_k = 1$. According to the discussion at the end of paragraph 5 it follows that $N_1(\bar{e}_i) = 0$ so that $A = N_0(\bar{e}_i) + N_1(\bar{e}_i) = N_1(1 - \bar{e}_i) + N_1(\bar{e}_i)$. Thus the resolution of $\bar{e}_i$ according to (13) is $1 \cdot \bar{e}_i + 0 \cdot (1 - \bar{e}_i)$ so that, by Theorem 18, $C(\bar{e}_i) = A$. Hence $\bar{e}_i$ is interchangeable with every b and $\bar{e}_i \in Z$. But $\bar{e}_1, \cdots, \bar{e}_k$ are linearly independent and the dimension of Z is greater than or equal to k.

Conversely, suppose that a belongs to Z. Then every polynomial in a belongs to Z and, in particular (because of (12)), the elements $e_1, \cdots, e_\mu$ of the resolution (13). We shall show that $e_1, \cdots, e_\mu$ are linear aggregates of $\bar{e}_1, \cdots, \bar{e}_k$ and hence that a is such an aggregate. It follows from this that the dimension of Z is k.

Let $e \in Z$ be a unit element. Its resolution according to (13) is $1 \cdot e + 0 \cdot (1 - e)$. Thus $C(e) = N_0(e) + N_1(e)$. Since $e \in Z$, $C(e) = A$, i.e., $N_0(e) + N_1(e) = A$. N_0 and N_1 are of course orthogonal. If we resolve the algebras N_0 and N_1 into their irreducible components: $N_0 = B_1' + \cdots + B_\alpha'$, $N_1 = B_1'' + \cdots + B_\beta''$, then we have the resolution $A = B_1' + \cdots + B_\alpha' + B_1'' + \cdots + B_\beta''$ which must be a permutation of $A = B_1 + \cdots + B_k$. But the modulus e of N_1 is the sum of the moduli of $B_1'', \cdots, B_\beta''$ (cf. the corresponding situation relative to A in the preceding theorem). Hence e is the sum of the moduli of some of the B_i, that is, of some of the $\bar{e}_i$. Hence e is a linear aggregate of $\bar{e}_1, \cdots, \bar{e}_k$ as asserted above.

By Theorem 19, A is irreducible (that is, $k = 1$) when and only when the dimension of Z is unity, that is, when A consists of the elements $\lambda 1$ alone.

10. The trace of a is a (real) number defined by the equation

$$\text{trace } a = \Delta \text{ trace } \|T_{\alpha\beta}(a)\| = \Delta \sum_{\alpha=1}^{N} T_{\alpha\alpha}(a).$$

As we know, the trace is independent of the selection of the basis of A (cf. the end of paragraph 8). The normalization factor Δ is some positive number which we shall select later.

[13] Cf. M. Zorn, Hamb. Abh., vol. 8, 1930, p. 123. This follows from the distributive law of multiplication alone.

It is clear that the trace is linear in a. The relation (which we do not prove)

$$O_{[a,\ b,\ c]} = O_b(O_aO_c - O_cO_a) - (O_aO_c - O_cO_a)O_b$$

implies the same relation for the corresponding matrices. From this it follows that trace $[a, b, c] = 0$ (since the trace of every matrix commutator vanishes). This is noteworthy because of its analogy with certain formulas of the quantum mechanical matrix-theory, but it will not be used in the sequel. We now consider other properties of the trace which are essential for the development of the quantum mechanical calculus.

THEOREM 20. *If e and e' are two unresolvable unit elements, there exists a resolution of 1 in pairwise orthogonal unresolvable unit elements: $e_1 + \cdots + e_\gamma = 1$, where $e_1 = e$ and $\gamma \geqq 2$, so that*

$$(32) \qquad \begin{cases} e' = \alpha e_1 + (1 - \alpha)e_2 + x & (0 \leqq \alpha \leqq 1,\ x \in M^{12}), \\ x^2 = \alpha(1 - \alpha)\ (e_1 + e_2) & \text{(Cf. Theorem 12).} \end{cases}$$

Proof: We make the resolution $e' = x_0 + x_1 + x_1$, where $x_0 \in N_0(e)$, $x_1 \in N_1(e)$, and $x_1 \in N_1(e)$. Since e is unresolvable, it follows by Theorem 11 that $x_1 = \alpha e$. We resolve x_0 in the algebra $N_0(e)$ according to (13): $x_0 = \xi_\nu e_2 + \cdots + \xi_\gamma e_\gamma$. (We begin with the index 2 for formal reasons.) Since we can represent each e_ν as a sum of pairwise orthogonal unit elements (this sum being an element of $N(e_\nu)$ so that the unit elements for distinct ν are mutually orthogonal), we can arrive at the same representation with unresolvable unit elements. So we may assume that the preceding e_ν are unresolvable, but then the ξ_ν need no longer be distinct. The elements $e_2, \cdots, e_\gamma$ are all orthogonal to e_1 and their sum is the modulus $1 - e$ of $N_0(e) = N_1(1 - e)$. Hence if $e_1 = e$ then $e_1 + \cdots + e_\gamma = 1$ is a resolution of 1 of the desired sort. Since $N_1(e) = N_1(e_1) = M^{12} + \cdots + M^{1\gamma}$ (cf. the analogous argument in the proof of Theorem 9) we have

$$e' = \alpha e_1 + \xi_2 e_2 + \cdots + \xi_\gamma e_\gamma + x_2 + \cdots + x_\gamma,$$

where $x_2 \in M^{12}, \cdots, x_\gamma \in M^{1\gamma}$. By Theorem 12 we have that $x_\nu^2 = \eta_\nu(e_1 + e_\nu)$ and therefore that

$$e'^2 = (\alpha^2 + \eta_2 + \ldots + \eta_\gamma)\, e_1 + (\xi_2^2 + \eta_2)\, e_2 + \ldots + (\xi_\gamma + \eta_\gamma)\, e_\gamma + (\alpha + \xi_2)\, x_2 + \ldots + (\alpha + \xi_\gamma)\, x_\gamma + \sum_{\nu<\nu'} x_\nu x_{\nu'} \qquad (\nu, \nu' = 2, \ldots, \gamma).$$

As was the case with e', the terms of the summation belong to $M^{\nu\nu'}$ $(\nu, \nu' = 2, \cdots, \gamma;\ \nu < \nu')$, while the other terms in this equation belong respectively to $M^{11}, M^{22}, \cdots, M^{\gamma\gamma}, M^{12}, \cdots, M^{1\gamma}$. Since $e'^2 = e'$, therefore $x_\nu x_{\nu'} = 0$. With the help of formulas (22 c, d), (23) and (24) one may readily calculate that $x_\nu(x_\nu x_{\nu'}) = \frac{1}{4}\eta_\nu x_{\nu'}$ (represent x_ν and $x_{\nu'}$ as linear aggregates of the bases $s_i^{1\nu}$ and $s_j^{1\nu'}$). Hence either $x_{\nu'} = 0$ or $\eta_\nu = 0$ so that $x_\nu^2 = 0$ and $x_\nu = 0$. Hence at most one x_ν is different from zero and, by a permutation of the elements $e_2, \cdots, e_\gamma$, we

may take it to be x_2. Consequently $\eta_3 = \cdots = \eta_\gamma = 0$, $x_3 = \cdots = x_\gamma = 0$, and, since $e'^2 = e'$,

$$(33) \qquad \begin{cases} \alpha^2 + \eta_2 = \alpha, \xi_2^2 + \eta_2 = \xi_2, \xi_3^2 = \xi_3, \ldots, \xi_\gamma^2 = \xi_\gamma, \\ \alpha + \xi_2 = 1 \text{ for } x_2 \neq 0. \end{cases}$$

By (33) every ξ_ν is either zero or unity for $\nu = 3, \cdots, \gamma$. A solution remains a solution if a ξ_ν with value unity is given the value zero instead. If $\xi_\nu = 1$, then e_ν and $e' - e_\nu$ are two orthogonal unit elements provided that $e' - e_\nu \neq 0$. Since e' is unresolvable, it must be the case that $e' = e_\nu$. By a permutation of the elements $e_2, e_3, \cdots, e_\gamma$ it can be arranged that $e' = e_2$. This is in agreement with (32) when $\alpha = 0$ and $x = 0$. If, on the contrary, every ξ_ν is zero for $\nu = 3, \cdots, \gamma$, then we may draw the following conclusion from (33): If $x = 0$, then e' has one of the values $e_1, e_2, e_1 + e_2$, or 0. But it is obvious that 0 cannot be a value of e'. Neither can $e_1 + e_2$ because of our hypothesis of unresolvability. Hence e is either e_1 or e_2. This agrees with (32) in the case where α is 1 or 0 and $x = 0$. But if $x \neq 0$, then $\xi_2 = 1 - \alpha$, $\eta_2 = \alpha(1 - \alpha) > 0$ so that $0 < \alpha < 1$. This agrees with (32) in the case where α is neither 0 nor 1.

THEOREM 21. *If e and e' are non-orthogonal unit elements,* trace $(ee') > 0$.

Proof: By Theorem 10 we can represent e and e' as sums of unresolvable unit elements. Consequently it is sufficient to consider the case where e and e' are unresolvable. By (32) it follows that

$$ee' = \alpha e_1 + \tfrac{1}{2} x, \text{ where } x \epsilon M^{12}, x^2 = \alpha (1 - \alpha).$$

If we now select the basis of A defined in Theorem 14, then it follows by (22) and the definition of trace that

$$\text{trace } e_\rho = \Delta \left(1 + \tfrac{1}{2} \sum_{\rho \neq \sigma} \chi_{\rho\sigma}\right) > 0, \quad \text{trace } s_\mu^{\rho\sigma} = 0 \quad (\rho, \sigma = 1, \ldots, \gamma).$$

Hence if $\alpha > 0$, then trace $(ee') = \alpha/2$ trace $(e_1) > 0$. But since $0 \leqq \alpha \leqq 1$, therefore the only alternative is $\alpha = 0$, which in turn implies $x^2 = 0$, $x = 0$, and therefore $ee' = 0$.

THEOREM 22. *If A is irreducible then Δ can be given such a positive value that* trace $(e) = 1$ *for every unresolvable unit element e.*

Proof: For every resolution $e_1 + \cdots + e_\gamma$ of 1 in pairwise orthogonal unresolvable unit elements it follows by the trace formulas in the proof of Theorem 21 and the remarks at the end of paragraph 7 that

$$\text{trace } e_\rho = \Delta \left(1 + \frac{(\gamma - 1)\chi}{2}\right) = \frac{\Delta N}{\gamma}, \quad \text{trace } s_\mu^{\rho\sigma} = 0.$$

Thus γ, χ, and all trace e_ρ have the same values for every such resolution having the same $e_1 = e$. For any other unresolvable unit element e' it follows by (32) that trace e' = trace e_1 = trace e since trace $x = 0$ as all trace $s_\mu^{1\,2} = 0$. There-

fore every unresolvable unit element e has the same trace. Consequently γ and χ always have the same value. We may therefore set

$$\Delta = \frac{1}{1 + \frac{(\gamma - 1)\chi}{2}} = \frac{\gamma}{N} > 0 \tag{34}$$

and arrive at the result[14] that trace $e = 1$.

We shall always select Δ in this way if A is irreducible. We now formulate two immediate consequences.

THEOREM 23. *The numbers γ and χ defined at the end of paragraph 7 for an irreducible A are uniquely determined.*

Proof: Note the remark in the proof of Theorem 22.

THEOREM 24. *If A is irreducible a unit element e may be represented (in the sense of Theorem 10) as a sum of unresolvable unit elements. The number of addends is then always equal to* trace e.

Proof: The theorem follows immediately from Theorem 22.

The physical meaning of an r-number algebra can be determined from Theorems 6, 21, and 22. The elements $a, b, \cdots$ correspond to the measurable quantities of a quantum system. The possible values of an element a arise from its resolution (13): $a = \xi_1 e_1 + \cdots + \xi_\mu e_\mu$ and are the proper values $\xi_1, \cdots, \xi_\mu$. If the value of a is measured and found to be ξ_ν then the *state* of the quantum system is described by the unit element e_ν. In the state e the expectation of an element b is $\frac{\text{trace } (eb)}{\text{trace } e}$. If $\eta_1 f_1 + \ldots + \eta_j f_j$ is the resolution of b by (13), then the probability that the measured value of b be η_i is $\frac{\text{trace } (ef_i)}{\text{trace } e}$. The case where e is unresolvable corresponds to a measurement which leads to a "reiner Fall."

Part III. Determination of all r-Number Algebras

11. We shall now seek all the solutions of the equations of condition (23), (24), and (25). These solutions will determine all r-number algebras which we shall assume throughout to be irreducible. We shall proceed stepwise in that we shall successively consider the cases $\gamma = 1, 2, 3$, and $\gamma \geqq 4$.

The analysis starts with the linear basis of A described in Theorem 14 together with the multiplication rules (22). But before we actually begin, we remark that if the resolution theorem 14 is applicable to an r-number algebra then the condition $\chi_{\rho\sigma} = \chi > 0$ for irreducibility is not only necessary (as we already know) but also sufficient. Under this assumption every invariant subalgebra of A is either 0 or A. This means that if any $x \neq 0$ of A is given, all elements of A may be obtained from it by forming suitable linear aggregates of expressions

[14] This conclusion is valid because of the previous unique determination of γ and χ.

of the form $(((xa)b)\cdots)k$, where $a, b, \cdots, k$ belong to A. For the formulas (18) and (19) show that in any case the elements $x_{\rho\sigma}$ can be so obtained. If some $x_{\rho\rho} \neq 0$ then we get an e_ρ, and if some $x_{\rho\sigma} \neq 0$ for $\rho \neq \sigma$ then we get an $x_{\rho\sigma}^2 \neq 0$. Therefore we can get an element $e_\rho + e_\sigma$ and from it we get $(e_\rho + e_\sigma)e_\rho = e_\rho$. Thus an e_ρ can be obtained in any case and from it we can get every element $e_\rho a_{\rho\tau} = \frac{1}{2} a_{\rho\tau}$ for $\rho \neq \tau$. Since we may select $a_{\rho\tau} \neq 0$ it follows that $a_{\rho\tau}^2 \neq 0$; this leads to $e_\rho + e_\tau$ and to $(e_\rho + e_\tau)e_\tau = e_\tau$. Hence we have all e_ρ and with them all $e_\rho \cdot 2s_\mu^{\rho\sigma} = s_\mu^{\rho\sigma}$ for $\rho \neq \sigma$. This completes the proof.

12. $\gamma = 1$. In this case $A = M^{11}$ and $e_1 = 1$ so that A consists merely of the elements $\lambda 1$. Since $\lambda 1 \pm \mu 1 = (\lambda \pm \mu)1$, $\lambda(\mu 1) = (\lambda\mu)1$, and $(\lambda 1)\,(\mu 1) = (\lambda\mu)1$, it follows that A is isomorphic with the algebra of real numbers.

13. $\gamma = 2$.[15] In the basis of Theorem 14 we replace e_1 and e_2 by $e_1 + e_2 = 1$ and $e_1 - e_2 = s_{\chi+1}$, and we write $s_1, \cdots, s_\chi$ in place of $s_1^{12}, \cdots, s_\chi^{12}$. We observe that $N = \chi + 2$. This leads to the basis $1, s_1, \cdots, s_{N-1}$ of A and the multiplication rules

$$s_\mu^2 = 1,\ s_\mu s_\nu = 0 \text{ for } \mu \neq \nu.$$

Since $\chi > 0$ it follows that $N \geqq 3$.

Conversely, A is an r-number algebra for each $N \geqq 3$. This may be seen by verifying (I) and (III). In fact, if $a = \alpha 1 + \alpha_1 s_1 + \cdots + \alpha_{N-1} s_{N-1}$, then $a^2 = (\alpha^2 + \alpha_1^2 + \cdots + \alpha_{N-1}^2)1 + 2\alpha\alpha_1 s_1 + \cdots + 2\alpha\alpha_{N-1} s_{N-1}$. If $a \neq 0$ the coefficient of 1 is greater than zero, so that if $a^2 + b^2 + c^2 + \cdots = 0$ then $a = b = c = \cdots = 0$. But a^2 has the form $\beta 1 + \gamma a$ so that from the definition of 1 and the commutativity of multiplication it follows that $(a^2 b)a = a^2(ba)$.

14. $\gamma = 3$. It is now necessary to solve the equations (23) and (24). In this case (24) assumes the form

$$\sum_{\nu=1}^{\chi} (A_{\nu\lambda\mu}^{123} A_{\nu\lambda'\mu'}^{123} + A_{\nu\lambda\mu'}^{123} A_{\nu\lambda'\mu}^{123}) = \tfrac{1}{2}\,\delta_{\lambda\lambda'}\,\delta_{\mu\mu'}.$$

If we define χ matrices $B_1, \cdots, B_\chi$ by the equations

$$B_\lambda = \|B_{\lambda,\mu\nu}\|,\ \mu,\nu = 1, \ldots, \chi; \qquad B_{\lambda,\mu\nu} = 2A_{\nu\lambda\mu}^{123}, \tag{35}$$

then[16]

$$B_\lambda^* B_{\lambda'} + B_{\lambda'}^* B_\lambda = 2\delta_{\lambda\lambda'}\,1. \tag{36}$$

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[15] This case was first discussed by M. Zorn to whom the authors are indebted also for valuable assistance.

[16] We denote the transposed matrix of B by B^*: $B = \|b_{\alpha\beta}\|$ and $B^* = \|b_{\beta\alpha}\|$. 1 is the unit matrix $1 = \|\delta_{\alpha\beta}\|$.

One can effect changes in the $A^{1\,2\,3}_{\nu\lambda\mu}$ (that is, the B_μ) by altering the bases of M^{22}, M^{13}, and M^{23}. These latter changes are unessential and, by Theorem 13 and (21), the transformations are

$$(37)\qquad (s^{1\,2}_\mu)' = \sum_{\nu=1}^{\chi} \omega_{\mu\nu} s^{1\,2}_\nu, \qquad (s^{1\,3}_\mu)' = \sum_{\nu=1}^{\chi} \varphi_{\mu\nu} s^{1\,3}_\nu, \qquad (s^{2\,3}_\mu)' = \sum_{\nu=1}^{\chi} \psi_{\mu\nu} s^{2\,3}_\nu,$$

where the matrices $\|\omega_{\mu\nu}\|$, $\|\varphi_{\mu\nu}\|$, and $\|\psi_{\mu\nu}\|$ are orthogonal. By (22d) the elements $A^{1\,2\,3}_{\nu\lambda\mu}$ transform according to the relation

$$(38)\qquad (A^{1\,2\,3}_{\nu\lambda\mu})' = \sum_{\nu',\lambda',\mu'=1}^{\chi} \omega_{\mu\mu'}\, \varphi_{\lambda\lambda'}\, \psi_{\nu\nu'}\, A^{1\,2\,3}_{\nu'\lambda'\mu'},$$

or, in terms of the B_μ,

$$(39)\qquad B'_\lambda = \sum_{\lambda'=1}^{\chi} \varphi_{\lambda\lambda'}\, U^* B_{\lambda'} W,$$

where $U = \|\omega_{\mu\nu}\|$, $V = \|\varphi_{\mu\nu}\|$, and $W = \|\psi_{\mu\nu}\|$ are unitary matrices. One sees immediately that for every solution of (36) there is another resulting from (39). We shall regard such solutions as being not essentially distinct.

The solution of (36) was obtained in a well-known paper[17] by A. Hurwitz who showed in particular that χ must be 1, 2, 4, or 8. Since a part of Hurwitz's investigation (unimportant for his main purpose but important for our discussion in paragraph 15) contains an essential error,[18] it is convenient to carry out the entire argument independently.

If in (36) $\lambda = \lambda'$, then $B^*_\lambda B_\lambda = 1$ so that all B_λ are orthogonal. If we set $C_\lambda = B^*_\chi B_\lambda$, $\lambda = 1, \cdots, \chi - 1$, then the C_λ are also all orthogonal. If $\lambda \neq \lambda'$ then (36) may be written in the form $B^*_\lambda B_{\lambda'} = -B^*_{\lambda'} B_\lambda$ or $B^*_{\lambda'} B_\lambda\, B^*_{\lambda'} B_\lambda = -1$. Hence if $\lambda' = \chi$ ($\lambda \neq \lambda' = \chi$) then $C^2_\lambda = -1$ and if $\lambda, \lambda' \neq \chi$ ($\lambda \neq \lambda'$) then $(C^*_{\lambda'} C_\lambda)^2 = -1$ since $C^*_{\lambda'} C_\lambda = B^*_{\lambda'} B_\lambda$. Or again, $C^*_{\lambda'} \cdot C_\lambda C^*_{\lambda'} \cdot C_\lambda = -1$ while $C^*_{\lambda'} \cdot C^*_{\lambda'} C_\lambda \cdot C_\lambda = C^{*2}_{\lambda'} \cdot C^2_\lambda = 1$ so that $C_\lambda C^*_{\lambda'} = -C^*_{\lambda'} C_\lambda$, or (after multiplication on both sides by $C_{\lambda'}$) $C_{\lambda'} C_\lambda = -C_\lambda C_{\lambda'}$. Hence the C_λ's are orthogonal,

$$(40)\qquad C^2_\lambda = -1,\ C_\lambda C_{\lambda'} = -C_{\lambda'} C_\lambda \text{ for } \lambda \neq \lambda',$$

where $\lambda, \lambda' = 1, \cdots, \chi - 1$.

If B is any orthogonal matrix, then $B_1 = BC_1, \cdots, B_{\chi-1} = BC_{\chi-1}$,

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[17] "Über die Composition der quadratischen Formen von beliebig vielen Variablen," Goett. Nachr., 1898, p. 309.

[18] Namely, the assertion that for a given χ all solutions B'_λ may be obtained from one particular B_λ by means of the relation $B'_\lambda = U^* B_\lambda W$ (that is, without the use of φ in (39)), this relation assuming the form $C'_\lambda = W^* C_\lambda W$ in the case where $C_\lambda = B^*_\chi B_\lambda$. (Cf. below.) One sees that this is incorrect in the following way: in case χ is equal to 4 or 8, then $C_1 \cdot \cdots \cdot C_{\chi-1} = \pm 1$. This equation is invariant under the above transformation. Yet the left side has the opposite sign in the solution $B'_1 = -B$, $B'_2 = B_2, \cdots, B_\chi = B_\chi$. (Cf. footnote 17, pp. 313–315; our B, C, U, W are there denoted by A, B, P, Q.)

$B_\chi = B$ is the general solution of (36) in case $C_1, \cdots, C_{\chi-1}$ is the general solution of (40).

15. Let G_χ be that group whose modulus may be denoted by $\bar{1}$ and which possesses χ generators $-\bar{1}, \bar{\Gamma}_1, \cdots, \bar{\Gamma}_{\chi-1}$ which satisfy the following relations:

$$\begin{cases} (-\bar{1})^2 = \bar{1}, & (-\bar{1})\bar{\Gamma}_\lambda = \bar{\Gamma}_\lambda(-\bar{1}), \\ \bar{\Gamma}_\lambda^2 = -1, & \bar{\Gamma}_\lambda\bar{\Gamma}_{\lambda'} = (-\bar{1})\bar{\Gamma}_{\lambda'}\bar{\Gamma}_\lambda \text{ for } \lambda \neq \lambda'. \end{cases} \tag{41}$$

It is obvious that all the elements of this group may be written as $\bar{\Gamma}_{\lambda_1} \cdots \bar{\Gamma}_{\lambda_r}$ or as $(-\bar{1})\bar{\Gamma}_{\lambda_1} \cdots \bar{\Gamma}_{\lambda_r}$, $r = 0, \cdots, \chi-1$, $1 \leqq \lambda_1 < \cdots < \lambda_r \leqq \chi - 1$. Hence G_χ is finite and has 2^χ elements. The problem of solving (40) reduces to that of finding[19] all representations of G_χ by χ-dimensional orthogonal matrices in which $-\bar{1}$ is represented by the matrix -1. $\bar{\Gamma}_\lambda$ is then represented by C_λ, $\lambda = 1, \cdots, \chi - 1$.

We next determine the "irreducible" representations of G_χ (cf. footnote 19 for all concepts and theorems in the theory of representations), and, of course, merely the complex-unitary ones. Since no C_λ exist for $\chi = 1$ (the solution is simply that B_1 must be orthogonal) we assume that $\chi \geqq 2$.

The order of G_χ is 2^χ. The commutators of G_χ are $\bar{1}$ and $-\bar{1}$ and they form an invariant subgroup of order 2 so that the index of G_χ is $2^{\chi-1}$. An element of the group can be transformed only into itself and its product with -1, the latter being impossible for $\bar{1}$ and $-\bar{1}$ as well as for $\bar{\Gamma}_1 \cdots \bar{\Gamma}_{\chi-1}$, $(-\bar{1})\bar{\Gamma}_1 \cdots \bar{\Gamma}_{\chi-1}$ if χ is even. Hence 2 or 4 equivalence classes each contain one element according as χ is odd or even, while all others contain two elements. Their number is therefore $2^{\chi-1} + 1$ or $2^{\chi-1} + 2$. (The reader may complete the simple discussion of G_χ.) Hence G_χ has $2^{\chi-1} + 1$ or $2^{\chi-1} + 2$ inequivalent irreducible representations of which $2^{\chi-1}$ are of order 1. These are the commutative representations. But since every commutator is $\bar{1}$ or $-\bar{1}$, and is therefore representable by the matrices 1 or -1, these commutative representations are exactly those in which $-\bar{1}$ is represented by 1. The representations which are of interest to us are therefore those of order $\neq 1$. From the above discussion it follows that there are one or two such representations.

The orders of all irreducible representations are divisors of 2^χ and the sum of the squares of these orders is 2^χ, of which the representations of order 1 contribute $2^{\chi-1} \cdot 1^2 = 2^{\chi-1}$. The sum of the squares of the orders of the remaining representations is $2^\chi - 2^{\chi-1} = 2^{\chi-1}$. If χ is odd there is only one such representation and its order is $2^{\frac{\chi-1}{2}}$. If χ is even there are two and their orders are divisors of 2^χ. Hence their orders are powers of 2 and the sum of the squares of their orders can be $2^{\chi-1}$ only if each of them is $2^{\frac{\chi-2}{2}}$.

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[19] The G. Frobenius—I. Schur "theory of representations," which is prominent in what follows, may be found in I. Schur's treatise "Neue Begrundung der Theorie der Gruppencharaktere," Berliner Ber., 1905, p. 406.

The (real) representation we are seeking must be made up of these irreducible (complex) representations and its order χ must be some multiple of the common degrees, i.e., $2^{\frac{\chi-1}{2}}$ or $2^{\frac{\chi-2}{2}}$. It follows immediately that $\chi = 2, 4$, or 8. All these values are even, $\chi/2^{\frac{\chi-2}{2}} = 2, 2$, or 1, and our representation consists of 2, 2, or 1 irreducible parts. But since we wish to form a real representation from complex representations, we must keep the reality conditions in mind.[20]

A complex irreducible representation can have in this regard one of three properties: 1) it may be equivalent to a real representation, 2) it may be equivalent to no real representation but instead be equivalent to its own complex-conjugate representation, or 3) it may be inequivalent to its complex-conjugate representation. In order to distinguish these cases one must form the sum of the traces (characteristics) of the matrices representing the squares of all the elements of the group. According as this sum is greater than, equal to, or less than 0 the situation is described by 1), 3), or 2).

In G_χ the square of every element is $\bar{1}$ or $-\bar{1}$. Suppose α of the elements have $\bar{1}$ for their square and suppose β of the elements have $-\bar{1}$ for their square. Then $\alpha + \beta = 2^\chi$. The matrix representations of these squared elements are 1 or -1 and their trace is $2^{\frac{\chi-2}{2}}$ or $-2^{\frac{\chi-2}{2}}$. The sum in question is therefore $2^{\frac{\chi-2}{2}}(\alpha - \beta)$ and our criterion reduces merely to the questions whether α is greater than, equal to, or less than β. Enumeration shows that the relations $=$, $<$, and $>$ hold for the values 2, 4, and 8 respectively of χ. Hence the cases 3), 2), and 1) exist. For $\chi = 2$ each of our two irreducible representations is inequivalent to its own complex-conjugate. Hence each is equivalent to the other so that we may describe the two irreducible representations as complex-conjugate to one another. The real representation we are seeking must therefore contain each of them exactly once. For $\chi = 4$ each irreducible representation is in case 2) and the real representation sought must contain each of them an even number of times. Hence it contains one of them twice but does not contain the other. For $\chi = 8$ each irreducible representation can be described as real so that both may be used as solutions. We may summarize our discussion by saying that for $\chi = 2$ there is one solution, and for $\chi = 4$ or 8 there are two inequivalent solutions.

Since $C_1 \cdots C_{\chi-1}$ is interchangeable with all C_λ (since χ is even), it must be of the form $\xi 1$ in each irreducible representation, and since its square is 1, it must be ± 1. Since the solution for $\chi = 4$ consists of two equivalent irreducible matrices and since the solution for $\chi = 8$ is irreducible, we have for $\chi = 4$ or 8 that

$$C_1 \cdots C_{\chi-1} = \pm 1. \tag{42}$$

[20] Cf. G. Frobenius and I. Schur, Berliner Ber., 1906, p. 186.

Since an equivalence transformation does not alter this relation, while the transformation $C_1' = -C_1$, $C_2' = C_2$, $\cdots$, $C_{\chi-1}' = C_{\chi-1}$ interchanges the $+$ and $-$ signs in it, it follows that this transformation carries one solution over into the other. (Cf. footnote 18.)

If, for $\chi = 2$, we mean by s_λ simply $s_\lambda = 1$, and for $\chi = 4$ or 8 we mean the same or $s_1 = -1$, $s_2 = \cdots = s_\chi = 1$, then the most general solution C_λ' arises from a solution C_λ by the transformation $C_\lambda' = s_\lambda W^* C_\lambda W$, where W is orthogonal. Hence $B_\lambda' = s_\lambda B W^* C_\lambda W$ (the relation also holds for $\lambda = \chi$ if we set $C_\chi = 1$: $B_\chi' = s_\chi B W^* W = B$). With the help of (39) it follows that $U = WB^*$ and $\varphi_{\mu\nu} = s_\mu \delta_{\mu\nu}$.

To sum up: Solutions exist for $\chi = 1, 2, 4$, and 8, but for each of these values there is essentially only one solution in the sense of (37), (38), and (39). In particular we can set U and V equal to 1 for $\chi = 1$, we can set $V = 1$ for $\chi = 2$, and we can set $V = 1$ or $V = \|\bar{s}_\mu \delta_{\mu\nu}\|$ ($\bar{s}_1 = -1$, $\bar{s}_2 = \cdots = \bar{s}_\chi = 1$) for $\chi = 4$ or 8.

16. Having shown that there is only one solution, we may better find it in another way. (36) may also be reformulated (and this was the original formulation by A. Hurwitz). In fact, (36) is equivalent to

$$(43)\qquad \begin{cases} (x_1^2 + \ldots + x_\chi^2)(y_1^2 + \ldots + y_\chi^2) = z_1^2 + \ldots + z_\chi^2, \\ z_\lambda = \sum_{\mu,\nu=1}^{\chi} B_{\lambda,\mu\nu} x_\mu y_\nu \qquad (\lambda = 1, \ldots, \chi) \end{cases}$$

($x_1, \cdots, x_\chi$ and $y_1, \cdots, y_\chi$ being two sets of independent variables). The solution may be found by the well-known substitution

$$(44)\qquad \begin{cases} X = x_1 1 + x_2 i_2 + \cdots + x_\chi i_\chi,\ Y = y_1 1 + y_2 i_2 + \cdots + y_\chi i_\chi, \\ Z = z_1 1 + z_2 i_2 + \cdots z_\chi i_\chi, \\ Xy = Z \end{cases}$$

in case $i_1 = 1$, $i_2, \cdots, i_\chi$ constitute the basis of the following hypercomplex number systems: for $\chi = 1$, the real numbers; for $\chi = 2$, the ordinary complex numbers; for $\chi = 4$, the quaternions;[21] for $\chi = 8$, the so-called Cayley numbers or quasi-quaternions.[22]

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[21] Quaternions satisfy the relations $i_2^2 = i_3^2 = i_4^2 = -1$; $i_2 i_3 = -i_3 i_2 = i_4$; $i_3 i_4 - i_4 i_3 = i_2$; $i_4 i_2 = -i_2 i_4 = i_3$.

[22] These were introduced by Cayley, Corresp. Math. Phys., vol. 1, 1843, p. 452. (Published by P. H. Fuss). A summary, and at the same time historically exhaustive, description of them has been given by L. E. Dickson, Annals. of Math., vol. 20, 1918, p. 155 and particularly pp. 157–159. A detailed discussion of their algebraic properties has been given by M. Zorn, cf. footnote 13. Quasi-quaternions are defined by $i_2^2 = \ldots = i_8^2 = -1$ and each of the systems i_2, i_3, i_5; i_3, i_4, i_6; i_4, i_5, i_7; i_5, i_6, i_8; i_6, i_7, i_2; i_7, i_8, i_3; i_8, i_2, i_4 (together with $i_1 = 1$) has the formal properties of quaternions. They form a non-associative system of quantities.

If we denote the complex-conjugate of such a hypercomplex number $X = x_1 1 + x_2 i_2 + \cdots + x_\chi i_\chi$ by $\bar{X} = x_1 1 - x_2 i_2 - \cdots - x_\chi i_\chi$ and its real part by $R(X) = x_1$, then $R(X\bar{\imath}_\mu) = x_\mu$, $\mu = 1, \cdots, \chi$. Consequently, if we note that $R(XY) = R(YX)$,

$$2A_{\nu\lambda\mu}^{123} = B_{\lambda,\mu\nu} = R(i_\mu i_\nu \bar{\imath}_\lambda) = R(i_\nu \bar{\imath}_\lambda i_\mu). \tag{45}$$

By (45) it is possible to calculate these $A_{\nu\lambda\mu}^{123}$ with the help of the multiplication tables of the above hypercomplex number systems, but we shall not go into this. The general solution $(A_{\nu\lambda\mu}^{123})'$ arises from (45) by (38) where we can keep the restriction mentioned at the end of paragraph 15.

It follows from the rules (22) together with (43) or (45) that the algebra A is isomorphic with a system of matrices $A, B, C, \cdots$ of order 3, wherein the operations $a \pm b$, λa, and ab correspond to the matrix operations $A \pm B$, λA, and $\frac{A \times B + B \times A}{2}$ ($A \times B$ is the usual matrix multiplication). Again, e_ρ corresponds to that matrix which has the $\rho\rho$-element 1 but otherwise consists of zeros, and $s_\mu^{\rho\sigma}$ ($\rho < \sigma$) corresponds to that matrix which has i_μ for its $\rho\sigma$-element and $\bar{\imath}_\mu$ for its $\sigma\rho$-element but otherwise consists of zeros. Consequently A is the algebra of all third order Hermitian matrices, $\|X_{\rho\sigma}\|$, where $\rho, \sigma = 1, 2, 3$ and $X_{\rho\sigma} = \bar{X}_{\sigma\rho}$ (the diagonal elements $X_{\rho\rho}$ being real numbers), with the above addition and multiplication rules. Hence the matrix elements for $\chi = 1, 2, 4$, or 8 respectively are real numbers, complex numbers, quaternions, and quasi-quaternions and $N = \gamma + \frac{\gamma(\gamma-1)}{2}\chi = 3 + 3\chi = 6, 9, 15, 27$.

17. Conversely, if A is one of the above algebras, we shall ascertain whether or not it is an r-number algebra. Since we can form the system of Hermitian matrices described in the last paragraph for every order (and not merely of order three as was done there), and since we shall need the more general matrices later on, we generalize the problem as follows: A consists of all Hermitian matrices of order γ subject to the above addition and multiplication rules and having real numbers, complex numbers, quaternions, and quasi-quaternions respectively as elements. We provisionally assume merely that $\gamma \geqq 3$ (since $\gamma = 1$ or 2 leads merely to special cases of the results obtained in paragraphs 12 and 13).

It is obvious that it is necessary to verify only (I) and either (II) or (III). (I) follows from the fact that if $A = \|X_{\rho\sigma}\|$, then $\text{trace}(A^2) = \sum_{\rho,\sigma=1}^{\gamma} |X_{\rho\sigma}|^2$ and hence is real and, for $A \neq 0$, positive. (II) is evident for the first three cases ($\chi = 1, 2$, and 4) since the matrix elements belong to associative number systems so that $A^2, A^3, \cdots$ result from our multiplication AB just as with the usual matrix multiplication $A \times B$ (cf. footnote 1). It remains to consider only the fourth case ($\chi = 8$). We shall soon see that there is no solution for $\gamma \geqq 4$ so

that only the case $\gamma = 3$ is in question. Instead of proving (III) we could prove its generalization (4) (which leads to (III) if one replaces a, b, c, u by a, a, a, b) which has the advantage of being linear in all four variables a, b, c, u. It suffices for this purpose to substitute all combinations of the 27 elements of the linear basis. This would lead actually to $27^4 = 531{,}441$ combinations on the basis of the multiplication rules and footnote 22. Yet this number may be reduced by a few devices so that the investigation is possible without too many possibilities. It results that (4) holds in all cases. We omit the proof which may be found in a subsequent paper of A. A. Albert in this journal. Thus for $\gamma = 3$ the fourth case also leads to a solution.

18. $\gamma \geqq 4$. Since $N_1(e_1 + e_2 + e_3) = \sum_{\rho, \sigma = 1}^{3} M^{\rho\sigma}$ is a subalgebra of A which has the same χ as A but for which $\gamma = 3$, we again have $\chi = 1, 2, 4$, or 8. We shall examine these cases successively.

$\gamma \geqq 4$, $\chi = 8$. If in (25) we set $\rho, \sigma, \tau, \upsilon$ respectively equal to 1, 2, 3, and 4, and introduce the following matrices (as in (35)):

$$\tilde{B}_\lambda = \| \tilde{B}_{\lambda, \mu\nu} \|, \tilde{B}_{\lambda, \mu\nu} = 2A^{123}_{\mu\nu\lambda}; \tilde{C}_\lambda = \| \tilde{C}_{\lambda, \mu\nu} \|, \tilde{C}_{\lambda, \mu\nu} = 2A^{124}_{\nu\mu\lambda};$$

$$\tilde{D}_\lambda = \| \tilde{D}_{\lambda, \mu\nu} \|, \tilde{D}_{\lambda, \mu\nu} = 2A^{134}_{\lambda\mu\nu}; \tilde{E}_\lambda = \| \tilde{E}_{\lambda, \mu\nu} \|, E_{\lambda, \mu\nu} = 2A^{234}_{\lambda\nu\mu}.$$

then it follows from (25) that

$$\tilde{C}_\iota \tilde{E}^*_\kappa = \tilde{D}_\kappa \tilde{B}^*_\iota, \tag{46}$$

and all of these matrices are orthogonal. (The orthogonality follows from (24), with either $\lambda = \lambda'$ or $\mu = \mu'$, together with (23).) If now we replace ι by ι' in (46) and form the transposed matrices, we have

$$\tilde{E}_\kappa \tilde{C}^*_{\iota'} = \tilde{B}_{\iota'} \tilde{D}^*_\kappa.$$

Multiplication of (46) on the right by this last equation gives

$$\tilde{C}_\iota \tilde{C}^*_{\iota'} = \tilde{D}_\kappa \tilde{B}^*_\iota \tilde{B}_{\iota'} \tilde{D}^*_\kappa.$$

The left side of this equation is independent of κ so that

$$\tilde{D}_\kappa \tilde{B}^*_\iota \tilde{B}_{\iota'} \tilde{D}^*_\kappa = \tilde{D}_{\kappa'} \tilde{B}^*_\iota \tilde{B}_{\iota'} \tilde{D}^*_{\kappa'}, \qquad \tilde{D}^*_{\kappa'} \tilde{D}_{\kappa'} \tilde{B}^*_\iota \tilde{B}_{\iota'} = \tilde{B}^*_\iota \tilde{B}_{\iota'} \tilde{D}^*_{\kappa'} \tilde{D}_{\kappa'}.$$

Hence every $\tilde{B}^*_\iota \tilde{B}_{\iota'}$ commutes with every $\tilde{D}^*_{\kappa'} \tilde{D}_{\kappa'}$.

We now have, with regard to (23),

$$\tilde{D}_{\lambda, \mu\nu} = 2A^{134}_{\lambda\mu\nu} = 2A^{413}_{\nu\lambda\mu}.$$

Hence in the subalgebra $N_1(e_1 + e_3 + e_4) = \sum_{\rho, \sigma = 1, 3, 4} M^{\rho\sigma}$ of A the matrix $\tilde{D}_\lambda = \| \tilde{D}_{\lambda, \mu\nu} \|$ plays the same rôle as did the matrix $B_\lambda = \| B_{\lambda, \mu\nu} \|$ in paragraphs 14 to 17 in the r-number algebra with $1 = e_1 + e_2 + e_3$. (The indices 1, 2, 3 are

replaced by 4, 1, 3.) The matrices $\tilde{D}_\chi^* \tilde{D}_\kappa$, $\kappa = 1, \cdots, \chi - 1$, therefore play the rôle of the C_κ and therefore form, for our $\chi = 8$, an irreducible system (cf. paragraph 15). Since every $\tilde{B}_\iota^* \tilde{B}_{\iota'}$ is interchangeable with every $\tilde{D}_\chi^* \tilde{D}_\kappa$, they all have the form $\xi_{\iota\,\iota'} 1$. But with regard to (23),

$$\tilde{B}_{\lambda,\,\mu\nu} = 2A_{\mu\,\nu\,\lambda}^{1\,2\,3} = 2A_{\nu\,\lambda\,\mu}^{2\,3\,1}.$$

Hence in the subalgebra $N_1(e_1 + e_2 + e_3) = \sum_{\rho,\,\sigma\,=\,1}^{3} M^{\rho\sigma}$ of A the matrix $\tilde{B}_\lambda = \|\tilde{B}_{\lambda,\,\mu\nu}\|$ plays the same rôle as did the matrix $B_\lambda = \|B_{\lambda,\,\mu\nu}\|$ in paragraphs 14 to 17 in the r-number algebra with $1 = e_1 + e_2 + e_3$. (The indices 1, 2, 3 are replaced by 2, 3, 1.) The matrices $\tilde{B}_\chi^* \tilde{B}_\kappa$, $\kappa = 1, \cdots, \chi - 1$, play the rôle of the C_κ and form, for our $\chi = 8$, an irreducible system. Therefore the form $\xi_{\chi\kappa} 1$ with the order $\chi = 8 > 1$ is impossible.

Thus in the above case there exists no solution.

19. $\gamma \geqq 4$, $\chi = 4$. In (45) we obtained a solution for $2A_{\nu\,\lambda\,\mu}^{1\,2\,3}$ in the form $R(i_\nu \bar{i}_\lambda\, i_\mu)$, where $i_1 = 1$, i_2, i_3, i_4 compose the basis of the system of quaternions. (Cf. footnote 21.) By (38) the most general solution results from it if we replace

$$i_\nu \text{ by } \sum_{\nu'\,=\,1}^{4} \psi_{\nu\nu'}\, i_{\nu'}, \qquad i_\mu \text{ by } \sum_{\mu'\,=\,1}^{4} \varphi_{\mu\mu'}\, i_{\mu'}, \qquad i_\lambda \text{ by } \sum_{\lambda'\,=\,1}^{4} \omega_{\lambda\lambda'}\, i_{\lambda'}$$

(we make no use of the restriction proved at the end of paragraph 15). But every orthogonal transformation $\sum_{\kappa'\,=\,1}^{4} \sigma_{\kappa\kappa'}\, i_{\kappa'}$ of the basis of the quaternion system may also be represented in the form $X' i_\kappa \bullet X''$, where X' and X'' represent certain quaternions of absolute value unity and $\bullet$ denotes nothing at all or else the sign for the complex conjugate.[23] The most general expression for $2A_{\nu\,\lambda\,\mu}^{1\,2\,3}$ is therefore (since $R(XY) = R(YX)$)

$$2A_{\nu\,\lambda\,\mu}^{1\,2\,3} = R(Y^{(1\,2\,3)}\, i_\nu^{(3\,1\,2)}_{\bullet}\, Y^{(2\,3\,1)}\, i_\lambda^{(1\,2\,3)}_{\bullet}\, Y^{(3\,1\,2)}\, i_\mu^{(2\,3\,1)}_{\bullet})$$

where the Y's are certain quaternions of absolute value unity and where the $\bullet$'s are independent of one another. Since $R(XY) = R(YX)$ it follows by (23) that this equation also holds for $2A_{\nu\,\lambda\,\mu}^{2\,3\,1}$ and $2A_{\nu\,\lambda\,\mu}^{3\,1\,2}$ if the indices 1, 2, and 3 are correspondingly permuted on the right side. Hence we may set

$$2A_{\nu\lambda\mu}^{\rho\sigma\tau} = R(Y^{(\rho\sigma\tau)}\, i_\nu^{(\tau\rho\sigma)}_{\bullet}\, Y^{(\sigma\tau\rho)}\, i_\lambda^{(\rho\sigma\tau)}_{\bullet}\, Y^{(\tau\rho\sigma)}\, i_\mu^{(\sigma\tau\rho)}_{\bullet}) \tag{47}$$

where the Y's are certain quaternions of absolute value unity and where the $\bullet$'s are independent of one another. Consequently we shall limit ourselves to the

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[23] We will write $\bullet = 0$ if $\bullet$ denotes nothing, and $\bullet = -$ if it denotes the complex conjugate. The representation of rotations by means of quaternions can be verified easily.

consideration of such values of ρ, σ, and τ which may be transformed by a cyclic permutation into a monotonically increasing sequence, and we shall call them allowable sequences. (We have treated the permutations of 1, 2, and 3 in the same manner. Since ρ, σ, τ or τ, σ, ρ is always allowed in this sense, it follows that the allowable $A^{\rho\sigma\tau}_{\lambda\mu\nu}$ are sufficient for the purpose of the multiplication rules in (22d).)

On the other hand the generality of (47) can be somewhat lessened in that we can subject every $M^{\rho\sigma}$ to an arbitrary transformation in the sense of Theorem 13, (21), and (37). This is exhibited in the formulas (47) for $2A^{\rho\sigma\tau}_{\nu\lambda\mu}$, $2A^{\sigma\tau\rho}_{\nu\lambda\mu}$, $2A^{\tau\rho\sigma}_{\nu\lambda\mu}$, or $2A^{\sigma\rho\tau}_{\nu\lambda\mu}$, $2A^{\rho\tau\sigma}_{\nu\lambda\mu}$, $2A^{\tau\sigma\rho}_{\nu\lambda\mu}$ as a transformation of the corresponding i_μ, i_λ, i_ν and hence as that of the i_κ with ${}^{(\sigma\tau\rho)}_{\bullet}$ or ${}^{(\rho\tau\sigma)}_{\bullet}$. If we transform all the $M^{\rho\sigma}$ simultaneously so that in $M^{\rho\sigma}$ i_κ goes over into $X_1^{(\rho\sigma)}\, i_\kappa^{(\rho\sigma)}\, X_2^{(\rho\sigma)}$, and if for the sake of simplicity we identify $\rho\sigma$ and $\sigma\rho$, then we have (since $\overline{XY} = \bar{Y}\bar{X}$)

(48)
$$\begin{cases} \text{(a)} \quad {}^{(\rho\sigma\tau)}_{\bullet} \text{ is replaced by } {}^{(\rho\sigma\tau)}_{\bullet}\,{}^{(\rho\tau)}_{\bullet}, \\ \text{(b)} \quad Y^{(\rho\sigma\tau)} \text{ is replaced by } X_{\mathrm{I}}^{(\rho\sigma)}\, Y^{(\rho\sigma\tau)}\, X_{\mathrm{II}}^{(\sigma\tau)}, \text{ where} \\ \qquad X_{\mathrm{I}}^{(\rho\sigma)} = X_2^{(\rho\sigma)} \text{ or } \bar{X}_{\mathrm{I}}^{(\rho\sigma)} \text{ when } {}^{(\sigma\tau\rho)}_{\bullet} = 0 \text{ or } \overline{}, \\ \qquad X_{\mathrm{II}}^{(\sigma\tau)} = X_{\mathrm{I}}^{(\sigma\tau)} \text{ or } \bar{X}_2^{(\sigma\tau)} \text{ when } {}^{(\tau\rho\sigma)}_{\bullet} = 0 \text{ or } \overline{}. \end{cases}$$

We now consider (25). (Since ρ, σ, τ; ρ, σ, υ; ρ, τ, υ; and σ, τ, υ are to be allowable, it must be the case that every ρ, σ, τ, υ goes over into a monotonically increasing sequence by a cyclic permutation and hence must be allowable.) If we substitute

$$R(XY) = R(YX), \quad \sum_{\kappa=1}^{4} R(X\, i_\kappa^{(1)}_{\bullet})\, R(Y\, i_\kappa^{(2)}_{\bullet}) = R(X\bar{Y}^{(1)}_{\bullet}\,{}^{(2)}_{\bullet})$$

in (47), we get

$$R([Y^{(\sigma\upsilon\rho)}\, i_\theta^{(\rho\sigma\upsilon)}\, Y^{(\upsilon\rho\sigma)}\, i_\iota^{(\sigma\upsilon\rho)}\, Y^{(\rho\sigma\upsilon)}] \cdot [Y^{(\upsilon\sigma\tau)}\, i_\lambda^{(\tau\upsilon\sigma)}\, Y^{(\sigma\tau\upsilon)}\, i_\kappa^{(\upsilon\sigma\tau)}\, Y^{(\tau\upsilon\sigma)}]^{\overline{(\upsilon\rho\sigma)}\,\overline{(\sigma\tau\upsilon)}})$$
$$= R([Y^{(\rho\tau\upsilon)}\, i_\kappa^{(\upsilon\rho\tau)}\, Y^{(\tau\upsilon\rho)}\, i_\theta^{(\rho\tau\upsilon)}\, Y^{(\upsilon\rho\tau)}] \cdot [Y^{(\tau\rho\sigma)}\, i_\iota^{(\sigma\tau\rho)}\, Y^{(\rho\sigma\tau)}\, i_\lambda^{(\tau\rho\sigma)}\, Y^{(\sigma\tau\rho)}]^{(\tau\upsilon\rho)}\,{}^{(\rho\sigma\tau)}_{\bullet}).$$

Since we are forming linear aggregates we can set a free quaternion variable Z_θ in place of i_θ and likewise free quaternion variables Z_ι, Z_λ, and Z_κ in place of i_ι, i_λ, i_κ. Then we can multiply out the parentheses on both sides of the equation using the law $\overline{(XY)} = \bar{Y}\bar{X}$. It can accordingly be seen that, because of the non-commutativity of quaternion multiplication, equality can exist only if variables which are in juxtaposition on one side are also in juxtaposition on the other side. Since Z_θ and Z_ι, as well as Z_λ and Z_κ, are in every case adjacent on the left side of the equation, they are again adjacent on the right side. Therefore $\overline{{}^{(\tau\upsilon\rho)}_{\bullet}\,{}^{(\rho\sigma\tau)}_{\bullet}}$ can not be equal to $\overline{}$, and hence ${}^{(\tau\upsilon\rho)}_{\bullet} \neq {}^{(\rho\sigma\tau)}_{\bullet}$. Similarly the invariable juxtaposition of Z_κ and Z_θ on the right side of the equation, as well as of Z_ι and Z_λ,

leads to the condition $^{(\upsilon\rho\sigma)}_{\bullet} \neq {}^{(\sigma\tau\upsilon)}_{\bullet}$. Now our equation assumes the form (since $R(XY) = R(YX)$)

$$R(Y^{(\tau\upsilon\sigma)}\, Y^{(\sigma\upsilon\rho)}\, i_\theta^{(\rho\sigma\upsilon)}\, Y^{(\upsilon\rho\sigma)}\, i_\iota^{(\sigma\upsilon\rho)}\, Y^{(\rho\sigma\upsilon)}\, Y^{(\upsilon\sigma\tau)}\, i_\lambda^{(\tau\upsilon\sigma)}\, Y^{(\sigma\tau\upsilon)}\, i_\kappa^{(\upsilon\sigma\tau)})$$
$$= R(Y^{(\tau\upsilon\rho)}\, i_\theta^{(\rho\tau\upsilon)}\, Y^{(\upsilon\rho\tau)}\, Y^{(\tau\rho\sigma)}\, i_\iota^{(\sigma\tau\rho)}\, Y^{(\rho\sigma\tau)}\, i_\lambda^{(\tau\rho\sigma)}\, Y^{(\sigma\tau\rho)}\, Y^{(\rho\tau\upsilon)}\, i_\kappa^{(\upsilon\rho\tau)}).$$

Again we can replace $i_\theta, i_\iota, i_\lambda, i_\kappa$ by $Z_\theta, Z_\iota, Z_\lambda, Z_\kappa$ and it results that

$$^{(\rho\sigma\upsilon)}_{\bullet} = {}^{(\rho\tau\upsilon)}_{\bullet},\ {}^{(\sigma\upsilon\rho)}_{\bullet} = {}^{(\sigma\tau\rho)}_{\bullet},\ {}^{(\tau\upsilon\sigma)}_{\bullet} = {}^{(\tau\rho\sigma)}_{\bullet},\ {}^{(\upsilon\sigma\tau)}_{\bullet} = {}^{(\upsilon\rho\tau)}_{\bullet}$$

in addition to the conditions

$$^{(\tau\upsilon\rho)}_{\bullet} \neq {}^{(\rho\sigma\tau)}_{\bullet},\ {}^{(\upsilon\rho\sigma)}_{\bullet} \neq {}^{(\sigma\tau\upsilon)}_{\bullet}$$

already obtained, so that

$$\text{(Y)} \qquad \begin{cases} Y^{(\tau\upsilon\sigma)}Y^{(\sigma\upsilon\rho)} = Y^{(\tau\upsilon\rho)}, & Y^{(\upsilon\rho\sigma)} = Y^{(\upsilon\rho\tau)}Y^{(\tau\rho\sigma)}, \\ Y^{(\rho\sigma\upsilon)}Y^{(\upsilon\sigma\tau)} = Y^{(\rho\sigma\tau)}, & Y^{(\sigma\tau\upsilon)} = Y^{(\sigma\tau\rho)}Y^{(\rho\tau\upsilon)}. \end{cases}$$

It is to be noted that we have required of the sequences ρ, σ, τ; ρ, σ, υ; ρ, τ, υ; and σ, τ, υ that each of them be transformable into a monotonically increasing sequence by a cyclic permutation, and of course the same must hold also of $\rho, \sigma, \tau, \upsilon$.

We now see that the four $\bullet$–equations simply comprise all possible subcases of $^{\alpha\epsilon\beta}_{\bullet} = {}^{\alpha\eta\beta}_{\bullet}$ for which α, ϵ, β and α, η, β are allowable sequences. The two next following $\bullet$-inequalities express together that $^{\alpha\epsilon\beta}_{\bullet} \neq {}^{\beta\eta\alpha}_{\bullet}$, where α, ϵ, β and β, η, α are arbitrary allowable sequences. By the operation of (48 *a*) we can, for every pair $\alpha > \beta$, make a $^{\alpha\epsilon\beta}_{\bullet}$ vanish or make a $^{\beta\eta\alpha}_{\bullet}$ equal to $\overline{\ }$.[24] From our $\bullet$-equations and $\bullet$-inequalities it follows that in any case every $^{\alpha\epsilon\beta}_{\bullet}$ will vanish and every $^{\beta\eta\alpha}_{\bullet}$ will be equal to $\overline{\ }$. Hence, for $\rho < \sigma < \tau$, $^{\rho\sigma\tau}_{\bullet}$ equals $\overline{\ }$, while $^{\sigma\tau\rho}_{\bullet}$ and $^{\tau\rho\sigma}_{\bullet}$ vanish.

The Y-equations all arise by cyclic permutations of the first which may also be written

$$Y^{(\tau\upsilon\sigma)} = Y^{(\tau\upsilon\rho)}\,\overline{Y^{(\sigma\upsilon\rho)}}.$$

If now $\bar{\upsilon}$ is the immediate successor of υ (that is, $\upsilon + 1$ for $\upsilon \neq \gamma$, 1 for $\upsilon = \gamma$), then we have

$$Y^{(\tau\upsilon\sigma)} = T^{(\tau\upsilon)}\,\overline{T^{(\sigma\upsilon)}},$$

where we define $T^{(\omega\upsilon)} = Y^{(\omega\upsilon\bar{\upsilon})}$ $(\omega \neq \upsilon)$ so long as the right side has meaning, and if it is meaningless $(\omega = \bar{\upsilon})$ then $T^{(\omega\upsilon)} = 1$. In this equation it is obvious that $\tau \neq \upsilon$; for $\tau \neq \bar{\upsilon}$ the equation follows from the preceding one with $\rho = \bar{\upsilon}$, and for $\tau = \bar{\upsilon}$ it is an identity. (Note that $T^{(\rho\sigma)}$ and $T^{(\sigma\rho)}$ are not identical.)

Among the operations of (48 *b*) we select

$$X_1^{(\alpha\beta)} = T^{(\beta\alpha)},\ X_2^{(\alpha\beta)} = \overline{T^{(\alpha\beta)}} \qquad (\alpha < \beta).$$

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[24] We must leave both alternatives open since, for example, $\alpha\epsilon\beta$ is never an allowable sequence for $\alpha = \gamma$ and $\beta = 1$, and the same holds for β, η, α when $\alpha = \beta + 1$.

(The left sides are symmetric in α and β while the right sides are not. Hence, for $\alpha > \beta$, we obtain valid formulas by interchanging α and β on the right sides.) If $\rho < \sigma < \tau$, then, by (48 b), $Y^{(\rho\sigma\tau)}$, $Y^{(\sigma\tau\rho)}$, $Y^{(\tau\rho\sigma)}$ all go over into 1 since $^{(\rho\sigma\tau)}_{\bullet} = \overline{}$ and $^{(\sigma\tau\rho)}_{\bullet} = {}^{(\tau\rho\sigma)}_{\bullet} = 0$. We have therefore made all the Y's simultaneously equal to 1, and if $\rho < \sigma < \tau$, then, by (47),

$$2A^{\rho\sigma\tau}_{\nu\lambda\mu} = R(i_\nu \bar{i}_\lambda i_\mu). \tag{49}$$

This result is exactly similar to that for $2A^{1\,2\,3}_{\nu\lambda\mu}$ in (45).

The multiplication rules (22) are accordingly uniquely determined by (49) and (23). Therefore in the preceding case there is at most one solution. But in any event a solution exists, namely, the Hermitian quaternion matrices of order γ (cf. paragraph 17). Hence this is the only solution.

20. $\gamma \geqq 4$, $\chi = 2$. In (45) we have obtained a solution $2A^{1\,2\,3}_{\nu\lambda\mu}$ in the form $R(i_\nu \bar{i}_\lambda i_\mu)$, where $i_1 = 1$ and $i_2 = i$ constitute the basis of the system of ordinary complex numbers. By (38) the most general solution arises if we replace

$$i_\nu \text{ by } \sum_{\nu'=1}^{2} \psi_{\nu\nu'} i_{\nu'}, \qquad i_\lambda \text{ by } \sum_{\lambda'=1}^{2} \varphi_{\lambda\lambda'} i_{\lambda'}, \qquad i_\mu \text{ by } \sum_{\mu'=1}^{2} \omega_{\mu\mu'} i_{\mu'}.$$

(We make no use of the restrictions proved at the end of paragraph 15). But every orthogonal transformation $\sum_{\kappa'=1}^{2} \sigma_{\kappa\kappa'} i_{\kappa'}$ of the basis of the complex number system may also be represented in the form $X i_\kappa \bullet$, where X is a complex number of absolute value unity and $\bullet$ represents nothing at all or else the sign for the complex conjugate. The most general expression for $2A^{1\,2\,3}_{\nu\lambda\mu}$ is therefore (note the commutativity of complex numbers)

$$2A^{1\,2\,3}_{\nu\lambda\mu} = R\,(Y^{(1\,2\,3)}\, i_\nu^{(3\,1\,2)}_{\bullet}\, i_\lambda^{(1\,2\,3)}_{\bullet}\, i_\mu^{(2\,3\,1)}_{\bullet}),$$

where $Y^{(1\,2\,3)}$ is a certain complex number of absolute value unity and where the $\bullet$'s are independent of one another. It follows by (23) that this equation also holds for $2A^{2\,3\,1}_{\nu\lambda\mu}$ and $2A^{3\,1\,2}_{\nu\lambda\mu}$ if the indices 1, 2, and 3 are correspondingly permuted on the right side, provided that we define $Y^{(123)} = Y^{(231)} = Y^{(312)}$. Therefore we may set

$$2A^{\rho\sigma\tau}_{\nu\lambda\mu} = R\,(Y^{(\rho\sigma\tau)}\, i_\nu^{(\tau\rho\sigma)}_{\bullet}\, i_\lambda^{(\rho\sigma\tau)}_{\bullet}\, i_\mu^{(\sigma\tau\rho)}_{\bullet}),\ Y^{(\rho\sigma\tau)} = Y^{(\sigma\tau\rho)} = Y^{(\tau\rho\sigma)}, \tag{50}$$

where the Y's are certain complex numbers of absolute value unity and where the $\bullet$'s are independent of one another. As in paragraph 19 we limit ourselves to such values of ρ, σ, τ which may be transformed into a monotonically increasing sequence by a cyclic permutation.

Again we must determine the influence of the basis transformations of the $M^{\rho\sigma}$ (in the sense of Theorem 13, (21) and (37)) on (50). This is exhibited in the formulas (47) for $2A^{\rho\sigma\tau}_{\nu\lambda\mu}$, $2A^{\sigma\tau\rho}_{\nu\lambda\mu}$, $2A^{\tau\rho\sigma}_{\nu\lambda\mu}$, or $2A^{\sigma\rho\tau}_{\nu\lambda\mu}$, $2A^{\rho\tau\sigma}_{\nu\lambda\mu}$, $2A^{\tau\sigma\rho}_{\nu\lambda\mu}$ as a transformation of the corresponding i_μ, i_λ, i_ν and hence as that of the i_κ with $^{(\sigma\tau\rho)}_{\bullet}$ or $^{(\rho\tau\sigma)}_{\bullet}$.

If we transform all the $M^{\rho\sigma}$ simultaneously so that i_κ in $M^{\rho\sigma}$ goes over into $X^{(\rho\sigma)}i_\kappa^{(\rho\sigma)}$, and if for the sake of simplicity we identify $\rho\sigma$ and $\sigma\rho$, then we have (note the commutativity of complex numbers)

$$(51)\quad \begin{cases} \text{(a)} & {}^{(\rho\sigma\tau)}_{\bullet} \text{ is replaced by } {}^{(\rho\sigma\tau)}_{\bullet}\,{}^{(\rho\tau)}_{\bullet}, \\ \text{(b)} & Y^{(\rho\sigma\tau)} \text{ is replaced by } Y^{(\rho\sigma\tau)}\,X^{(\sigma\tau)}\,{}^{(\tau\rho\sigma)}_{\bullet}\,X^{(\tau\rho)}\,{}^{(\rho\sigma\tau)}_{\bullet}\,X^{(\rho\sigma)}\,{}^{(\sigma\tau\rho)}_{\bullet}. \end{cases}$$

Further, since $R(X) = R(\bar{X})$ and $\overline{XY} = \bar{X}\bar{Y}$, we can immediately

$$(51)\quad \begin{cases} (c) \text{ alter all } {}^{(\rho\sigma\tau)}_{\bullet},\ {}^{(\sigma\tau\rho)}_{\bullet}, \text{ and } {}^{(\tau\rho\sigma)}_{\bullet} \text{ simultaneously,} \\ \quad \text{and replace } Y^{(\rho\sigma\tau)} \text{ by } \overline{Y^{(\rho\sigma\tau)}} \text{ in (50).} \end{cases}$$

By application of (51 c) we can simultaneously obtain ${}^{(\rho\sigma\tau)}_{\bullet} = {}^{-}$ for all ρ, σ, τ such that $\rho < \sigma < \tau$.

We now consider (25). We assume $\rho < \sigma < \tau < \upsilon$. Then it follows that ${}^{(\rho\sigma\tau)}_{\bullet} = {}^{(\rho\sigma\upsilon)}_{\bullet} = {}^{(\rho\tau\upsilon)}_{\bullet} = {}^{(\sigma\tau\upsilon)}_{\bullet} = {}^{-}$. If we substitute

$$\sum_{\kappa=1}^{2} R(Xi_\kappa^{(1)}_{\bullet})\,R(Yi_\kappa^{(2)}_{\bullet}) = R(X\bar{Y}^{(1)}_{\bullet}\,{}^{(2)}_{\bullet})$$

and the particular choice of some $\bullet$'s just now formulated into (50), we get

$$R(Y^{(\rho\sigma\upsilon)}\,Y^{(\sigma\tau\upsilon)}\,{}^{(\upsilon\rho\sigma)}_{\bullet}\,\bar{\imath}_\theta\,i_\iota\,{}^{(\sigma\upsilon\rho)}_{\bullet}\,i_\lambda{}^{(\tau\upsilon\sigma)}_{\bullet}\,{}^{(\upsilon\rho\sigma)}_{\bullet}\,i_\kappa{}^{(\upsilon\sigma\tau)}_{\bullet}\,{}^{(\upsilon\rho\sigma)}_{\bullet})$$
$$= R(Y^{(\rho\tau\upsilon)}\,Y^{(\rho\sigma\tau)}\,{}^{(\tau\upsilon\rho)}_{\bullet}\,i_\kappa{}^{(\upsilon\rho\tau)}_{\bullet}\,\bar{\imath}_\theta\,i_\iota{}^{(\sigma\tau\rho)}_{\bullet}\,{}^{(\tau\upsilon\rho)}_{\bullet}\,i_\lambda{}^{(\tau\rho\sigma)}_{\bullet}\,{}^{(\tau\upsilon\rho)}_{\bullet}).$$

Since we are forming linear aggregates we can set free complex variables Z_θ, Z_ι, Z_λ, Z_κ in place of i_θ, i_ι, i_λ, i_κ. The above equality can hold only if the two parentheses are equal or if one is equal to the conjugate of the other. But since $\overline{Z_\theta}$ occurs on both sides the latter possibility is excluded. Consequently it must be the case that

$${}^{(\sigma\upsilon\rho)}_{\bullet} = {}^{(\sigma\tau\rho)}_{\bullet}\,{}^{(\tau\upsilon\rho)}_{\bullet},\quad {}^{(\tau\upsilon\rho)}_{\bullet}\,{}^{(\upsilon\rho\sigma)}_{\bullet} = {}^{(\tau\rho\sigma)}_{\bullet}\,{}^{(\tau\upsilon\rho)}_{\bullet},\quad {}^{(\upsilon\sigma\tau)}_{\bullet}\,{}^{(\upsilon\rho\sigma)}_{\bullet} = {}^{(\upsilon\rho\tau)}_{\bullet}.$$

The first equation can also be written ${}^{(\tau\upsilon\rho)}_{\bullet} = {}^{(\sigma\tau\rho)}_{\bullet}\,{}^{(\sigma\upsilon\rho)}_{\bullet}$ so that if $\bar{\rho}$ is the immediate successor of ρ (that is, $\rho + 1$ inasmuch as $\rho \neq \gamma$ because of the condition $\rho < \sigma < \tau < \upsilon$)

$${}^{(\tau\upsilon\rho)}_{\bullet} = {}^{(\,(\tau\rho)\,)}_{\bullet}\,{}^{(\,(\upsilon\rho)\,)}_{\bullet},$$

where we define ${}^{(\,(\omega\rho)\,)}_{\bullet} = {}^{(\bar{\omega}\omega\rho)}_{\bullet}$ $(\omega \neq \rho)$ so long as the right side has meaning, and if it is meaningless $(\omega = \bar{\rho})$ then ${}^{(\,(\omega\rho)\,)}_{\bullet} = 0$.

For $\rho < \sigma < \tau$ it follows that ${}^{(\rho\sigma\tau)}_{\bullet} = {}^{-}$ and ${}^{(\sigma\tau\rho)}_{\bullet} = {}^{(\,(\sigma\rho)\,)}_{\bullet}\,{}^{(\,(\tau\rho)\,)}_{\bullet}$, where ${}^{(\,(\tau\rho)\,)}_{\bullet}$ is arbitrary. If for ${}^{(\,(\tau\rho)\,)}_{\bullet} = {}^{-}$ we apply the operation (51 c) and for ${}^{(\,(\tau\rho)\,)}_{\bullet} = 0$ we do nothing, then it results that ${}^{(\rho\sigma\tau)}_{\bullet} = \overline{{}^{(\,(\tau\rho)\,)}_{\bullet}}$ and ${}^{(\sigma\tau\rho)}_{\bullet} = {}^{(\,(\sigma\rho)\,)}_{\bullet}$, where ${}^{(\tau\rho\sigma)}_{\bullet}$ is arbitrary. If to this result we apply the operation (51 a) with ${}^{(\alpha\beta)}_{\bullet} = {}^{(\,(\alpha\beta)\,)}_{\bullet}$ $(\alpha > \beta)$, then it results that ${}^{(\rho\sigma\tau)}_{\bullet} = {}^{-}$ and ${}^{(\sigma\tau\rho)}_{\bullet} = 0$, where ${}^{(\tau\rho\sigma)}_{\bullet}$ is arbitrary. From these equations we have

$${}^{(\upsilon\rho\sigma)}_{\bullet} = {}^{(\tau\rho\sigma)}_{\bullet},\quad {}^{(\upsilon\sigma\tau)}_{\bullet}\,{}^{(\upsilon\rho\sigma)}_{\bullet} = {}^{(\upsilon\rho\tau)}_{\bullet}.$$

The first states that $^{(\omega\rho\sigma)}_{\bullet}$ (naturally $\rho < \sigma < \omega$) is independent of ω so that $^{(\omega\rho\sigma)}_{\bullet} = {}^{(((\rho\sigma)))}_{\bullet}$ and the second states that $^{(((\sigma\tau)))}_{\bullet}\ ^{(((\rho\sigma)))}_{\bullet} = {}^{(((\rho\tau)))}_{\bullet}$. Therefore $^{(((\rho\sigma)))}_{\bullet} = {}^{[\rho]}_{\bullet}\,{}^{[\sigma]}_{\bullet}$ and $^{(\omega\rho\sigma)}_{\bullet} = {}^{[\rho]}_{\bullet}\,{}^{[\sigma]}_{\bullet}$. Therefore $^{(\rho\sigma\tau)}_{\bullet} = \overline{\ }$, $^{(\sigma\tau\rho)}_{\bullet} = 0$, and $^{(\tau\rho\sigma)}_{\bullet} = {}^{[\rho]}_{\bullet}\,{}^{[\sigma]}_{\bullet}$. If for $^{[\rho]}_{\bullet} = \overline{\ }$ we here apply the operation (51 c) and for $^{[\rho]}_{\bullet} = 0$ we do nothing, then it results that $^{(\rho\sigma\tau)}_{\bullet} = \overline{{}^{[\rho]}_{\bullet}}$, $^{(\sigma\tau\rho)}_{\bullet} = {}^{[\rho]}_{\bullet}$, and $^{(\tau\rho\sigma)}_{\bullet} = {}^{[\sigma]}_{\bullet}$. If we apply the operation (51 a) with $^{(\alpha\beta)}_{\bullet} = {}^{[\beta]}_{\bullet}$ ($\alpha > \beta$), then we have $^{(\rho\sigma\tau)}_{\bullet} = \overline{\ }$ and $^{(\sigma\tau\rho)}_{\bullet} = {}^{(\tau\rho\sigma)}_{\bullet} = 0$.

In the second equation following (51 c) the coefficients in the parentheses must be equal to one another so that for $\rho < \sigma < \tau < \upsilon$ it follows that $Y^{(\rho\sigma\upsilon)}\ Y^{(\sigma\tau\upsilon)}\ {}^{(\upsilon\rho\sigma)}_{\bullet} = Y^{(\rho\tau\upsilon)}\ Y^{(\rho\sigma\tau)}\ {}^{(\tau\upsilon\rho)}_{\bullet}$ so that with the $\bullet$ just obtained, $Y^{(\rho\sigma\upsilon)}\ Y^{(\sigma\tau\upsilon)} = Y^{(\rho\tau\upsilon)}\ Y^{(\rho\sigma\tau)}$ and hence $Y^{(\rho\sigma\tau)} = Y^{(\sigma\tau\upsilon)}\ \overline{Y^{(\rho\tau\upsilon)}}\ Y^{(\rho\sigma\upsilon)}$. Consequently

$$Y^{(\rho\sigma\tau)} = T^{(\sigma\tau)}\ \overline{T^{(\rho\tau)}}\ T^{(\rho\sigma)},$$

where we set $T^{(\alpha\beta)} = Y^{(\alpha\beta\gamma)}$ ($\alpha < \beta$) so long as the right side has meaning, and if it is meaningless ($\beta = \gamma$) then $T^{(\alpha\beta)} = 1$. We now select from the operations of (51 b)

$$X^{(\alpha\beta)} = \overline{T^{(\alpha\beta)}} \qquad (\alpha < \beta).$$

(The left side is symmetric in α and β while the right side is not so that we obtain the formulas valid for $\alpha > \beta$ by interchange of α and β on the right side.) Then all $Y^{(\rho\sigma\tau)}$ go over into 1.

By (50) it follows that, for $\rho < \sigma < \tau$,

$$2A^{\rho\sigma\tau}_{\nu\lambda\mu} = R(i_\nu\ \overline{i_\lambda}\ i_\mu) \tag{52}$$

which is similar to (49) and to $2A^{1\,2\,3}_{\nu\,\lambda\,\mu}$ in (45).

Hence the multiplication rules (22) are uniquely determined by (52) and (23). There is therefore at most one solution in the preceding case. But a solution exists in any event, for example, the Hermitian complex matrices of order γ (cf. paragraph 17). Hence it is the only solution.

21. $\gamma \geqq 4$, $\chi = 1$. In this case it is not worth while to carry out the general theory of paragraphs 14 to 17. Since $\lambda = \mu = \nu = 1$ in $A^{\rho\sigma\tau}_{\nu\lambda\mu}$, therefore $2A^{\rho\sigma\tau}_{\nu\lambda\mu} = Y^{\rho\sigma\tau}$ and (24) leads to the result $Y^{\rho\sigma\tau} = \pm 1$. The basis transformations of $\mathfrak{M}^{\rho\sigma}$ in the sense of Theorem 13, (21), and (37) are $(s_1^{\rho\sigma})' = X^{\rho\sigma}\ s_1{}^{\rho\sigma}$ and $X^{\rho\sigma} = \pm 1$, where for the sake of simplicity we identify $\rho\sigma$ and $\sigma\rho$. Hence

$$Y^{(\rho\sigma\tau)} \text{ is replaced by } Y^{(\rho\sigma\tau)}\ X^{(\sigma\tau)}\ X^{(\tau\rho)}\ X^{(\rho\sigma)}. \tag{53}$$

In the preceding case it follows from (25) that

$$Y^{(\rho\sigma\upsilon)}\ Y^{(\sigma\tau\upsilon)} = Y^{(\rho\tau\upsilon)}\ Y^{(\rho\sigma\tau)}. \tag{54}$$

The relation

$$Y^{(\rho\sigma\tau)} = T^{(\sigma\tau)}\ T^{(\rho\tau)}\ T^{(\rho\sigma)}$$

follows from (54) in exactly the same way as did the analogous relation for the complex Y at the end of paragraph 20. We similarly find that every T is either a Y or 1 and hence is ± 1. Therefore we select from the operations of (53)

$$X^{(\alpha\beta)} = T^{(\alpha\beta)} \qquad (\alpha < \beta).$$

(With reference to the symmetry conditions in α and β, cf. the remark at the end of paragraph 20.) Then all $Y^{(\rho\sigma\tau)}$ go over into 1. Hence we have in general that

$$2A^{\rho\sigma\tau}_{\nu\lambda\mu} = 1. \tag{54}$$

The multiplication rules (22) are uniquely determined by (54) and (23). Hence in the preceding case there is at most one solution. But in any event a solution exists, for example, the Hermitian (real) matrices of order γ (cf. paragraph 17). Hence it is the only solution.

22. We now summarize our results.

In the course of the preceding investigations we have learned the properties of the following algebras:

α. $\mathfrak{R}$, the algebra of the real numbers in which $a \pm b$, λa, and ab are defined in the usual way.

β. $\mathfrak{S}_N$, $N = 1, 2, \cdots$, the algebra with the linear basis $1, s_1, \cdots, s_{N-1}$, in which $a \pm b$ and λa are defined in the usual way but in which ab is defined by

$$11 = 1,\ 1s_\mu = s_\mu, \qquad s_\mu s_\nu = \delta_{\mu\nu} 1.$$

γ. $\mathfrak{M}^\chi_\gamma$, $\chi = 1, 2, 4, 8$; $\gamma = 1, 2, \cdots$, the algebra of the Hermitian matrices of order γ in which the elements are real numbers, ordinary complex numbers, quaternions, and quasi-quaternions respectively for $\chi = 1, 2, 4$, and 8. $a \pm b$ and λa are defined in the usual way, but ab is defined by $\dfrac{a \times b + b \times a}{2}$, where $a \times b$ represents the usual matrix multiplication.

We can summarize the results of paragraphs 12 to 21 as follows:

FUNDAMENTAL THEOREM 2. *The only irreducible r-number algebras are the following:*

For $\gamma = 1$: $\mathfrak{R}$.

For $\gamma = 2$: *All* $\mathfrak{S}_N$, $N = 3, 4, \cdots$. ($\mathfrak{S}_1$ *is* $\mathfrak{R}$ *and* $\mathfrak{S}_2$ *is reducible.*)

For $\gamma = 3$: *All* $\mathfrak{M}^\chi_3$, $\chi = 1, 2, 4, 8$.

For $\gamma \geqq 4$: *All* $\mathfrak{M}^\chi_\gamma$, $\chi = 1, 2, 4$. (*All* $\mathfrak{M}^\chi_1$ *are* $\mathfrak{R}$; *the* $\mathfrak{M}^\chi_2$ *are respectively the* $\mathfrak{S}_{\chi+2}$; *the* $\mathfrak{M}^\chi_3$ *are already classified under* $\gamma = 3$; *the* $\mathfrak{M}^8_\gamma$, $\gamma \geqq 4$, *are not r-number algebras.*)

In connection with this result we mention the following circumstance: let A' be an associative, but not necessarily commutative algebra having the real

numbers as coefficients. Then $a \pm b$, λa, and $a \times b$ are the fundamental operations and all the usual rules of calculation are valid except for $a \times b = b \times a$. We now define $ab = \frac{a \times b + b \times a}{2}$ and we consider a subsystem A of A' which is closed for $a \pm b$, λa, and ab (but not necessarily for $a \times b$) and which is "formally real" ($a^2 = a \times a = aa$; cf. (I) in paragraph 1 and footnote 5). Since (II) is obviously satisfied (a^n is the same for ab and $a \times b$), A is an r-number algebra. By the theorem of Wedderburn[25] we could select a matrix algebra for A' without loss of generality. We say (in agreement with the terminology of footnote 1) that an r-number algebra which is isomorphic with such an A *results from quasi-multiplication.* As one of us has shown elsewhere (cf. footnote 1), essentially new results, in contrast with the present content of quantum mechanics, are to be expected only in such r-number algebras which do *not* result from quasi-multiplication.

Now all the r-number algebras of our list, with the exception of $\mathfrak{M}_3^8$, result from quasi-multiplication. This is obvious in the case of $\mathfrak{R}$ and the $\mathfrak{M}_\gamma^\chi$ ($\chi = 1, 2, 4$) so that only the $\mathfrak{S}_N$ need examination. But in any event the $\mathfrak{S}_N$ arise by the operation $ab = \frac{a \times b + b \times a}{2}$ from a system in which 1 is the modulus, $s_\mu^2 = 1$, and $s_\mu s_\nu = -s_\nu s_\mu (\mu \neq \nu, \mu, \nu = 1, \cdots, N-1)$. We now take the quaternion-like, non-commutative, but associative system $1, i, j, k$ with $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and form $N-1$ sets $1^{(1)}, i^{(1)}, j^{(1)}, k^{(1)}; \cdots; 1^{(N-1)}, i^{(N-1)}, j^{(N-1)}, k^{(N-1)}$ of this sort. The direct products $1 = 1^{(1)} \cdot 1^{(2)} \cdot \ \cdots \ \cdot 1^{(N-1)}$, $s_1 = j^{(1)} \cdot 1^{(2)} \cdot \ \cdots \ \cdot 1^{(N-1)}$, $s_2 = i^{(1)} \cdot j^{(2)} \cdot \ \cdots \ \cdot 1^{(N-1)}$, $\cdots$, $s_{N-1} = i^{(1)} \cdot i^{(2)} \cdot \ \cdots \ \cdot j^{(N-1)}$ perform the desired function.

On the other hand it can be shown that $\mathfrak{M}_3^8$ can not result from quasi-multiplication. The proof of this remarkable result was given by A. Albert, to whom we proposed the problem, and can be found in his next paper in this journal. Hence there is one single (irreducible) r-number algebra which does not arise from quasi-multiplication: $\mathfrak{M}_3^8$.

ROSTOCK, GERMANY, AND PRINCETON, N. J.

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[25] Proc. London Math. Soc., 1907, Vol. (2) 6, p. 99.

ANNALS OF MATHEMATICS
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ON A CERTAIN ALGEBRA OF QUANTUM MECHANICS[1]

BY A. ADRIAN ALBERT

(Received November 14, 1933)

1. **Introduction.** P. Jordan, J. von Neumann, and E. Wigner[2] have discussed certain linear real non-associative algebras of importance in quantum mechanics. Their algebras $\mathfrak{M}$ satisfy the ordinary postulates for addition, the commutative law for multiplication, and the distributive law, but they are non-associative.

In the paper quoted above it is shown that, with a single exception, every algebra satisfying the above postulates is equivalent to an algebra $\mathfrak{M}$ whose elements are ordinary real matrices $x, y, \cdots$ with products xy in $\mathfrak{M}$ defined by quasi-multiplication,

$$xy = \tfrac{1}{2}(x \cdot y + y \cdot x),$$

where $x \cdot y$ is the ordinary matrix product. This single exception is the algebra $\mathfrak{M}_3^8$ of all three rowed Hermitian matrices with elements in the real non-associative algebra C of Cayley numbers.

The algebras obtained by quasi-multiplication of real matrices were considered in earlier papers so that, as stated by the above authors, this seemingly exceptional case, if proved really exceptional, is the only algebra of the above type which could lead to any new form of quantum mechanics.

In the present paper I shall prove that $\mathfrak{M}_3^8$ is a new algebra and that it is not equivalent to any algebra obtained by quasi-multiplication of real matrices. Moreover, I shall show that the relation[3] $x(yx^2) = (xy)x^2$ is satisfied by $\mathfrak{M}_3^8$.

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$$(1)\quad \begin{cases} e_i^2 = -1, \quad e_j e_i = -e_i e_j \ (i, j = 1, \cdots, 7; i \neq j), \quad e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \\ e_1 e_6 = e_7, \quad e_2 e_5 = e_7, \quad e_2 e_4 = -e_6, \quad e_3 e_4 = e_7, \quad e_3 e_5 = e_6, \end{cases}$$

such that the algebras $(1, e_1, e_2, e_3)$, $(1, e_1, e_4, e_5)$, $(1, e_1, e_6, e_7)$, $(1, e_2, e_5, e_7)$, $(1, e_2, e_4, -e_6)$, $(1, e_3, e_4, e_7)$, $(1, e_3, e_5, e_6)$ are all equivalent to the algebra of real quaternions is an associative algebra and was first discussed by Cayley. However, if three e's are not in one of these seven algebras, then $e_a(e_b e_c) = -(e_a e_b)e_c$.

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such that the algebras $(1, e_1, e_2, e_3)$, $(1, e_1, e_4, e_5)$, $(1, e_1, e_6, e_7)$, $(1, e_2, e_5, e_7)$, $(1, e_2, e_4, -e_6)$, $(1, e_3, e_4, e_7)$, $(1, e_3, e_5, e_6)$ are all equivalent to the algebra of real quaternions is an associative algebra and was first discussed by Cayley. However, if three e's are not in one of these seven algebras, then $e_a(e_b e_c) = -(e_a e_b)e_c$.

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If $a = \lambda_0 + \sum_{i=1}^{7} \lambda_i e_i$, then we define

$$\bar{a} = \lambda_0 - \sum_{i=1}^{7} \lambda_i e_i, \quad T(a) = a + \bar{a} = \bar{a} + a = 2\lambda_0, \quad N(a) = a\bar{a} = \bar{a}a = \sum_{i=0}^{7} \lambda_i^2,$$

so that a satisfies $w^2 - T(a)w + N(a) = 0$. It is well known that $\overline{ab} = \bar{b}\bar{a}$, $N(ab) = N(a)\,N(b)$, $T(\lambda a + \mu b) = \lambda T(a) + \mu T(b)$ for any real numbers λ, μ and any elements a, b of C. Moreover, if a and b are *any two* elements of C, then the associative law holds for polynomials in a and b. In particular, $a(\bar{a}b) = (a\bar{a})b$. We now prove

THEOREM 1. *Let i be any unit e_j of C and let a and b be arbitrary elements of C. Then*[4]

$$(2) \quad i(ab)i = (ia)\,(bi), \qquad (iai)\,(ib) = -i(ab), \qquad (ai)\,(ibi) = -(ab)i.$$

The above relations are linear in a and b so that it suffices to prove (2) when a and b are units of C. If i, a, b are in the same quaternion system, then equations (2) obviously result by application of the associative law. Hence we suppose that i, a, b are not all in the same system. Without loss of generality we may take $i = e_1$, $a = e_2$, $b = e_4$. Then $i(ab)i = -i^2(ab) = e_2e_4 = -e_6$, $(ia)\,(bi) = e_3(-e_5) = -e_6$. Also, $(iai)\,(ib) = a(ib) = e_2e_5 = e_7$, $-i(ab) = e_1e_6 = e_7$. Finally, $(ac)\,(ibi) = (-e_3)\,(e_4) = -e_7$, $-(ab)i = i(ab) = -e_7$. This completes the proof of (2).

Let $e_0 = 1$ and write

$$a = \sum_{j=0}^{7} \lambda_j e_j, \qquad b = \sum_{k=0}^{7} \mu_k e_k,$$

so that $a\bar{b} = \sum_{j,k} (\lambda_j \mu_k) e_j \bar{e}_k$ and $T(a\bar{b}) = \sum_{j,k} \lambda_j \mu_k\, T(e_j \bar{e}_k)$.

If $j \neq k$, then $e_j\bar{e}_k \neq \pm 1$, so that $T(e_j\bar{e}_k) = 0$. If $j = k$, then $e_j\bar{e}_j = 1$. Hence

$$(3) \qquad \tfrac{1}{2}T(a\bar{b}) = \sum_{j=0}^{7} \lambda_j \mu_j.$$

If $u_0, \cdots, u_7$ represent any permutation of $\pm 1, \pm e_1, \cdots, \pm e_7$, then it is obvious that $\frac{1}{2}T(u_j\bar{u}_k) = 0$ or 1 according as $j \neq k$ or $j = k$, so that the value (3) is unaltered. We call such a substitution on the units e_i *an orthogonal transformation*. In particular, we have

THEOREM 2. *The transformations which replace $e_j (j = 0, \cdots, 7)$ by ie_j, e_ji, ie_ji, where $i \neq 1$ is any unit of C, are all orthogonal.*

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[4] We may evidently write iai, etc., without ambiguity since the associative law holds for any polynomials in only two quantities of C.

3. **The algebra $\mathfrak{M}_3^8$.** We shall consider the real algebra of twenty-seven units $\mathfrak{M}_3^8$ whose general element is the Hermitian matrix

$$A = \begin{pmatrix} \alpha & a & \bar{c} \\ \bar{a} & \beta & b \\ c & \bar{b} & \gamma \end{pmatrix}, \tag{4}$$

where α, β, γ are real variables and where a, b, c are variable Cayley numbers of our algebra C. We shall write

$$A = A[(\alpha, \beta, \gamma), (a, b, c)]. \tag{5}$$

If δ is any real number and if

$$B = A[(\lambda, \mu, \nu), (f, g, h)], \tag{6}$$

then we define

$$\begin{cases} A + B = A[(\alpha + \lambda, \beta + \mu, \gamma + \nu), (a + f, b + g, c + h)], \\ \delta A = A = [(\delta\alpha, \delta\beta, \delta\gamma)\ (\delta a, \delta b, \delta c)], \end{cases} \tag{7}$$

and

$$AB = BA = \tfrac{1}{2}(A \cdot B + B \cdot A) = A[(\xi, \eta, \zeta), (p, q, r)]. \tag{8}$$

Here $A \cdot B$ means the ordinary matrix product, so that

$$\begin{cases} \xi = \alpha\lambda + \tfrac{1}{2}T(a\bar{f}) + \tfrac{1}{2}T(\bar{c}h), & 2p = (\alpha + \beta) f + (\lambda + \mu) a + \bar{c}\bar{g} + \bar{h}\bar{b}, \\ \eta = \beta\mu + \tfrac{1}{2}T(b\bar{g}) + \tfrac{1}{2}T(\bar{a}f), & 2q = (\beta + \gamma) g + (\mu + \nu) b + \bar{a}\bar{h} + \bar{f}\bar{c}, \\ \zeta = \gamma\nu + \tfrac{1}{2}T(c\bar{h}) + \tfrac{1}{2}T(\bar{b}g), & 2r = (\gamma + \alpha) h + (\nu + \lambda) c + \bar{b}\bar{f} + \bar{g}\bar{a}. \end{cases} \tag{9}$$

We observe that ξ goes into η, η into ζ, ζ into ξ, p into q, q into r, r into p when we cyclically permute (α, β, γ), (a, b, c), (λ, μ, ν), (f, g, h) simultaneously.

4. **A property of $\mathfrak{M}_3^8$.** By the use of (9) we may easily compute $A^2 = A[(\alpha_0, \beta_0, \gamma_0), (a_0, b_0, c_0)]$, where

$$\begin{cases} \alpha_0 = \alpha^2 + N(a) + N(c), & a_0 = (\alpha + \beta) a + \bar{c}\bar{b}, \\ \beta_0 = \beta^2 + N(b) + N(a), & b_0 = (\beta + \gamma) b + \bar{a}\bar{c}, \\ \gamma_0 = \gamma^2 + N(c) + N(b), & c_0 = (\gamma + \alpha) c + \bar{b}\bar{a}. \end{cases} \tag{10}$$

Let $B = A[(\lambda, \mu, \nu), (0, 0, 0)]$, so that $AB = A[(\xi, \eta, \zeta), (p, q, r)]$, where

$$\xi = \alpha\lambda, \quad \eta = \beta\mu, \quad \zeta = \gamma\nu, \quad 2p = (\lambda + \mu)a, \quad 2q = (\mu + \nu)b, \quad 2r = (\nu + \lambda)c. \tag{11}$$

Similarly, $A^2B = A[(\xi_0, \eta_0, \zeta_0), (p_0, q_0, r_0)]$, where

$$\xi_0 = \alpha_0\lambda, \quad \eta_0 = \beta_0\mu, \quad \zeta_0 = \gamma_0\nu, \quad 2p_0 = (\lambda + \mu)a_0, \quad 2q_0 = (\mu + \nu)b_0, \quad 2r_0 = (\nu + \lambda)c_0. \tag{12}$$

We let $\alpha, \beta, \gamma, \lambda, \mu, \nu$ be independent real variables, we let a, b, c be independent variables in C, and we compute

$$\begin{cases} A(BA^2) = A[(\xi_1, \eta_1, \zeta_1), (p_1, q_1, r_1)], \\ (AB)A^2 = A[(\xi_2, \eta_2, \zeta_2), (p_2, q_2, r_2)], \end{cases} \tag{13}$$

where, by (9),

$$(14)\quad \begin{cases} \xi_1 = \alpha\xi_0 + \frac{1}{2}T(a\bar{p}_0) + \frac{1}{2}(\bar{c}r_0), \quad \xi_2 = \alpha_0\xi + \frac{1}{2}T(a_0\bar{p}) + \frac{1}{2}T(\bar{c}_0 r), \\ 2p_1 = (\alpha+\beta)\,p_0 + (\xi_0+\eta_0)\,a + \bar{c}\bar{q}_0 + \bar{r}_0\bar{b}, \quad 2p_2 = (\alpha_0+\beta_0)\,p + (\xi+\eta)\,a_0 + \bar{c}_0\bar{q} + \bar{r}\bar{b}_0. \end{cases}$$

It is sufficient to show that $\xi_1 = \xi_2$ and $p_1 = p_2$ in order to show that $A(BA^2) = (AB)A^2$. For, cyclic permutation of (α, β, γ), (a, b, c), (λ, μ, ν) induces cyclic permutation of (ξ_1, η_1, ζ_1), (p_1, q_1, r_1), (ξ_2, η_2, ζ_2), (p_2, q_2, r_2).

The element $2(a\bar{p}_0 - a_0\bar{p}) = (\lambda+\mu)(a\bar{a}_0 - a_0\bar{a})$ evidently has zero trace and $2(\bar{c}r_0 - \bar{c}_0 r) = (\nu+\lambda)(\bar{c}c_0 - \bar{c}_0 c)$ also has zero trace. Since $\alpha\xi_0 - \alpha_0\xi = \alpha(\alpha_0\lambda) - \alpha_0(\alpha\lambda) = 0$, we have shown that $\xi_1 - \xi_2 = 0$.

We next consider the difference $4(p_1 - p_2)$. Since $q = (\mu+\nu)b$ and $r = (\nu+\lambda)c$, we have

$$(15)\quad \begin{cases} 2[(\bar{c}\bar{q}_0 - \bar{c}_0\bar{q}) + (\bar{r}_0\bar{b} - \bar{r}\bar{b}_0)] = (\mu+\nu)(\bar{c}\bar{b}_0 - \bar{c}_0\bar{b}) + (\nu+\lambda)(\bar{c}_0\bar{b} - \bar{c}\bar{b}_0) \\ = (\lambda-\mu)(\bar{c}_0\bar{b} - \bar{c}\bar{b}_0) = (\lambda-\mu)[(\gamma+\alpha-\beta-\gamma)\,\bar{c}\bar{b} + (ab)\,\bar{b} - \bar{c}\,(ca)] \\ = (\lambda-\mu)(\alpha-\beta)\,\bar{c}\bar{b} + (\lambda-\mu)[N(b) - N(c)]a, \end{cases}$$

since $(ab)\bar{b} = N(b)a$, $\bar{c}(ca) = N(c)a$. Similarly, $p = (\lambda+\mu)a$, so that

$$(16)\quad 2(\alpha+\beta)\,p_0 - 2(\alpha_0+\beta_0)\,p = (\lambda+\mu)(\alpha+\beta)\,\bar{c}\bar{b} + (\lambda+\mu)[(\alpha+\beta)^2 - (\alpha_0+\beta_0)]\,a.$$

Finally, $\xi = \alpha\lambda$, $\eta = \beta\mu$, so that

$$(17)\quad 2[(\xi_0+\eta_0)\,a - (\xi+\eta)\,a_0] = 2[(\alpha_0\lambda + \beta_0\mu) - (\alpha\lambda+\beta\mu)(\alpha+\beta)]\,a - 2(\alpha\lambda+\beta\mu)\,\bar{c}\bar{b}.$$

But $(\lambda-\mu)(\alpha-\beta) + (\lambda+\mu)(\alpha+\beta) = 2(\alpha\lambda+\beta\mu)$, so that the total coefficient of $\bar{c}\bar{b}$ in $4(p_1 - p_2)$ is zero. Also,

$$\begin{aligned} &(\lambda+\mu)(\alpha+\beta)^2 - 2(\alpha+\beta)(\alpha\lambda+\beta\mu) + 2(\alpha_0\lambda+\beta_0\mu) - (\lambda+\mu)(\alpha_0+\beta_0) \\ &= (\alpha+\eta)[(\lambda-\mu)(\beta-\alpha)] + (\lambda-\mu)(\alpha_0-\beta_0) \\ &= (\lambda-\mu)[-(\alpha+\beta)(\alpha-\beta) + \alpha^2 - \beta^2 + N(c) - N(b)] = (\lambda-\mu)[N(c) - N(b)], \end{aligned}$$

so that, by (15), the total coefficient of a is zero. We have proved the relation $p_1 = p_2$ and

THEOREM 3. *For every A of $\mathfrak{M}_3^8$ and every $B = A\,[(\lambda, \mu, \nu), (0, 0, 0)]$, we have $(AB)A^2 = A(BA^2)$.*

5. The principal theorem in $\mathfrak{M}_3^8$. Consider the correspondence

$$(18)\qquad A = A[(\alpha, \beta, \gamma), \quad (a, b, c)] \leftrightarrow A^* = A[(\alpha, \beta, \gamma), (a', b'', c''')],$$

where $a' = iai$, $b'' = ib$, $c''' = ci$, and i is any unit of C different from unity. The correspondence between A and A^* is evidently preserved under addition and real scalar multiplication. To prove that it is preserved under multiplica-

tion, and hence that it is an automorphism of $\mathfrak{M}_3^8$, we need to show that $(AB)^* = A^*B^*$. We have

$$(AB)^* = A[(\xi, \eta, \zeta), (p', q'', r''')],$$
$$A^*B^* = A[(\xi^*, \eta^*, \zeta^*), (p^*, q^*, r^*)].$$

But, by Theorem 2, the above transformation on the units of C is orthogonal and hence preserves the value of $T(a\bar{f})$, $T(\bar{a}f)$, $T(b\bar{g})$, $T(\bar{b}g)$, $T(c\bar{h})$, $T(\bar{c}h)$. Using (9) we immediately have $\xi = \xi^*$, $\eta = \eta^*$, $\zeta = \zeta^*$.

To show that $p' = p^*$ we merely note that $(\bar{c}\bar{g})' = i(\bar{c}\bar{g})i = (\bar{c}\bar{g})^* = (\bar{c}\bar{\imath})\,(\bar{\imath}\bar{g}) = (i\bar{c})\,(\bar{g}i) = i(\bar{c}\bar{g})i$ by Theorem 1. Similarly, $(\bar{h}\bar{b})' = (\bar{h}\bar{b})^*$. Also, $q'' = q^*$ since $(\bar{a}\bar{h})'' = i(\bar{a}\bar{h})$ and $(\bar{a}\bar{h})^* = (\bar{\imath}\bar{a}\bar{\imath})\,(\bar{h}\bar{\imath}) = (i\bar{a}i)\,(i\bar{h}) = i(\bar{a}\bar{h}) = (\bar{a}\bar{h})''$ by Theorem 1. Finally, $(\bar{b}\bar{f})''' = (\bar{b}\bar{f})i$ and $(\bar{b}\bar{f})^* = (i\bar{b})\,(i\bar{f}i) = -(\bar{b}i)\,(i\bar{f}i) = (\bar{b}\bar{f})'''$ by Theorem 1, so that $r''' = r^*$.

We may now prove

LEMMA 1. *Let $F(B, D, \cdots, K)$ be a polynomial in the variables $B, D, \cdots, K$ in $\mathfrak{M}_3^8$, let F have real coefficients and be linear in B, and let $F = 0$ for any real matrix B of $\mathfrak{M}_3^8$. Then $F = 0$ for all $B, \cdots, K$ of $\mathfrak{M}_3^8$.*

In particular, $F = 0$ for $B = A[(0, 0, 0), (0, 0, 1)]$. But the above automorphism carries B into $B^* = B[(0, 0, 0), (0, 0, i)]$ and $D, \cdots, K$ into arbitrary $D^*, \cdots, K^*$. Hence $F = P$ for any such B^*. But every element c of algebra C is a linear combination, with real coefficients, of the quantities $1, e_1, \cdots, e_7$, and F is linear in B so that $F = 0$ for $B = A[(0, 0, 0), (0, 0, c)]$. By cyclically permuting the quantities in $A[(\lambda, \mu, \nu), (f, g, h)]$ we show that $F = 0$ for any $A[(0, 0, 0), (0, b, 0)]$, $A[(0, 0, 0), (a, 0, 0)]$. But $F = 0$ for $B = A[(\alpha, \beta, \gamma), (0, 0, 0)]$, so that $F = 0$ for $B = [(\alpha, \beta, \gamma), (a, b, c)]$.

We next prove

LEMMA 2. *Let $A^* = U \cdot A \cdot U^{-1}$, where*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad U' = U^{-1}$$

is a real orthogonal matrix, and the product $U \cdot A \cdot U^{-1}$ signifies ordinary matrix multiplication. Then A^* is in $\mathfrak{M}_3^8$, and the correspondence $A \leftrightarrow A^*$ generates an automorphism of $\mathfrak{M}_3^8$.

For obviously A^* is Hermitian when A is Hermitian, and U is orthogonal, so that A^* is in $\mathfrak{M}_3^8$. Moreover, $(A + B)^* = A^* + B^*$, $(\delta A)^* = \delta A^*$, while

$$(AB)^* = U \cdot \left(\frac{A \cdot B + B \cdot A}{2}\right) \cdot U^{-1} = \frac{U \cdot (A \cdot B) \cdot U^{-1}}{2} + \frac{U \cdot (B \cdot A) \cdot U^{-1}}{2}.$$

But $U \cdot (A \cdot B) \cdot U^{-1} = A^* \cdot B^*$ since U is a real matrix, and ordinary multiplication of matrices of Cayley numbers, in which only two factors are not real, is associative. Hence $(AB)^* = \frac{1}{2}(A^* \cdot B^* + B^* \cdot A^*) = A^*B^*$ and the above lemma is proved.

We now prove that it is sufficient to show that $F = 0$ when $B = A[(\lambda, \mu, \nu),$

$(0, 0, 0)]$ in order to prove Lemma 1. In fact, the above automorphism carries $B_1 = A[(0, 1, -1), (0, 0, 0)]$ into $B_1^* = A[(0, 0, 0), (0, 0, 1)]$. Hence if $F = 0$ for $B = B_1$ then $F = 0$ for $B = B_1^*$. But then, as in the proof of Lemma 1, $F = 0$ for any $B = A[(0, 0, 0), (a, b, c)]$. Since we are assuming $F = 0$ for $B = A[(\lambda, \mu, \nu), (0, 0, 0)]$, we have $F = 0$ for any B.

We consider, in particular, the case where $F \equiv (AB)A^2 - A(BA^2)$. The above argument and Theorem 3 lead to the

PRINCIPAL THEOREM. *For every A and B of $\mathfrak{M}_3^8$ we have $(AB)A^2 = A(BA^2)$, so that $\mathfrak{M}_3^8$ is an r-number system.*[5]

6. **The exceptional property of $\mathfrak{M}_3^8$.** P. Jordan, J. von Neumann, and E. Wigner proved[2] that every algebra satisfying their system of postulates, except $\mathfrak{M}_3^8$, is equivalent to a linear set $\mathfrak{N}$ of a real total matric algebra $\mathfrak{M}$, where multiplication AB in $\mathfrak{N}$ is defined by $AB = \frac{1}{2}(A \cdot B + B \cdot A)$ and where the products $A \cdot B$, $B \cdot A$ are defined by the multiplication law in $\mathfrak{M}$. We here prove

THEOREM 5. *There exists no $\mathfrak{N}$ equivalent to $\mathfrak{M}_3^8$. Thus the algebra $\mathfrak{M}_3^8$ has a unique place among the r-number systems.*

For let $\mathfrak{N}$ be equivalent to $\mathfrak{M}_3^8$. Write

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_i = \begin{pmatrix} 0 & e_i & 0 \\ \bar{e}_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_j = \begin{pmatrix} 0 & 0 & e_j \\ 0 & 0 & 0 \\ \bar{e}_j & 0 & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_k \\ 0 & \bar{e}_k & 0 \end{pmatrix},$$

$(i, j, k = 0, \cdots, 7)$

where $e_0, \cdots, e_7$ constitute a basis of the Cayley algebra C. Then $J_1, J_2, J_3, E_i, F_j, G_k$ form a basis of $\mathfrak{M}_3^8$. Let $K_1, K_2, K_3, P_i, Q_j, R_k$ form the corresponding basis of $\mathfrak{N}$.

The matrices J_1, J_2, J_3 are indempotent matrices so that K_1, K_2, K_3 are also idempotent. Since $(J_1 \cdot J_2 + J_2 \cdot J_1) = J_1 J_2 = 0$, $K_1 \cdot K_2 = K_1 \cdot K_2 + K_2 \cdot K_1 = 0$. But $K_1 \cdot (K_1 \cdot K_2 + K_2 \cdot K_1) \cdot K_1 = 2K_1 \cdot K_2 \cdot K_1 = 0$, and therefore $K_1 \cdot (K_1 \cdot K_2 + K_2 \cdot K_1) = K_1 \cdot K_2 = 0$. Similarly, $K_2 \cdot K_1 = K_1 \cdot K_3 = K_2 \cdot K_3 = K_3 \cdot K_1 = K_3 \cdot K_2 = 0$. Hence the idempotent matrices K_1, K_2, K_3 are supplementary. It is well known that a basis of $\mathfrak{M}$ may be so chosen that

$$K_1 = \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} L_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} L_3 & 0 \\ 0 & 0 \end{pmatrix},$$

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[5] For the definition of r-number system, see the preceding paper by P. Jordan, J. von Neumann, and E. Wigner.

where

$$L_1 = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and I_1, I_2, I_3 are identity matrices of r, s, t rows respectively.

The matrix $J_1 + J_2 + J_3$ is the three rowed identity matrix and is the identity element of $\mathfrak{M}_3^8$. Hence

$$K = K_1 + K_2 + K_3 = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix},$$

where L is the $(r + s + t)$-rowed identity matrix, then K is the identity element of $\mathfrak{N}$. But then if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is any quantity of $\mathfrak{N}$, we have

$$KA = \tfrac{1}{2}(K \cdot A + A \cdot K) = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}c & 0 \end{pmatrix} = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so that $b = c = d = 0$. Hence, without loss of generality, we may take the quantities of $\mathfrak{N}$ to be $(r + s + t)$-rowed square matrices and we may take $K = L$ as identity element. Note that $K_i = L_i$ $(i = 1, 2, 3)$.

If $E = \sum_{i=0}^{7} \lambda_i E_i$ in $\mathfrak{M}_3^8$, then (9) gives $2J_1E = 2J_2E = E$. But then if $P = \Sigma \lambda_i P_i$, we have $2L_1P = 2L_2P = P$. We write

$$P = (a_{ij})$$

and compute $L_i \cdot P + P \cdot L_i = P$, where $i = 1, 2$. Since

$$P = L_1 \cdot P + P \cdot L_1 = \begin{pmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = L_2 \cdot P + P \cdot L_2 = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 2a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix},$$

it follows that $a_{11} = a_{22} = a_{33} = a_{31} = a_{32} = a_{23} = a_{13} = 0$. Hence

$$P_i = \begin{pmatrix} 0 & p_i & 0 \\ p_{i0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where p_i is a matrix of r-rows and s-columns of real elements and where p_{i0} is a matrix of s-rows and r-columns. Similarly,

$$Q_j = \begin{pmatrix} 0 & 0 & q_j \\ 0 & 0 & 0 \\ q_{j0} & 0 & 0 \end{pmatrix}, \qquad R_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r_k \\ 0 & r_{k0} & 0 \end{pmatrix}.$$

By (9) we have

$$2E_iE_j = \begin{pmatrix} e_i\bar{e}_j + e_j\bar{e}_i & 0 & 0 \\ 0 & \bar{e}_ie_j + \bar{e}_je_i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which it is obvious that

$$2P_iP_j = \begin{pmatrix} p_ip_{j0} + p_jp_{i0} & 0 & 0 \\ 0 & p_{i0}p_j + p_{j0}p_i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But $\frac{1}{2}(e_i\bar{e}_i + e_i\bar{e}_i) = 1$ $(i = 0, \cdots, 7)$ and $\frac{1}{2}(\bar{e}_ie_i + \bar{e}_ie_i) = 1$. Hence we must have

$$p_ip_{i0} = I_1, \qquad p_{i0}p_i = I_2 \qquad (i = 0, \cdots, 7).$$

Similarly, $q_iq_{i0} = I_2$, $r_ir_{i0} = I_3$, etc. Moreover, if $i \neq j$, then $e_i\bar{e}_j + e_j\bar{e}_i = \bar{e}_ie_j + \bar{e}_je_i = 0$. Hence

$$p_ip_{j0} = -p_jp_{i0} \qquad (i, j = 0, \cdots, 7; i \neq j).$$

We next compute

$$2E_iG_j = \begin{pmatrix} 0 & 0 & e_ie_j \\ 0 & 0 & 0 \\ \bar{e}_j\bar{e}_i & 0 & 0 \end{pmatrix} = \lambda_{ij}F_k,$$

where $e_ie_j = \lambda_{ij}e_k$ is in C. Hence

$$2P_iR_j = \begin{pmatrix} 0 & 0 & p_ir_j \\ 0 & 0 & 0 \\ r_{j0}p_{i0} & 0 & 0 \end{pmatrix} = \lambda_{ij}Q_k,$$

from which

$$(19) \qquad p_ir_j = \lambda_{ij}q_k, \qquad r_{j0}p_{i0} = \lambda_{ij}q_{k0} \qquad (i, j = 0, \cdots, 7).$$

If, in particular, we take $i = 0$, then $e_0e_j = e_j = \lambda_{0j}e_k$, so that $\lambda_{0j} = 1$, $k = j$, and $p_0r_j = q_j$. Similarly, for $j = 0$, we obtain $q_i = p_ir_0$ and

$$p_0r_i = p_ir_0.$$

Multiplying on the left by p_{00} we obtain

$$r_i = (p_{00}p_i)r_0.$$

We use $q_i = p_ir_0$ in (19) and obtain

$$p_ir_j = p_i(p_{00}p_j)r_0 = \lambda_{ij}q_k = \lambda_{ij}p_kr_0.$$

Multiplying on the right by r_{00} we obtain

$$p_i(p_{00}p_j) = \lambda_{ij}p_k.$$

Let $u_i = p_i p_{00}$. Then u_i is an r-rowed real square matrix which is not zero for every value of i insomuch as $u_0 = I_1$. It follows that $p_i(p_{00}p_j)p_{00} = u_i u_j = \lambda_{ij} p_k p_{00} = \lambda_{ij} u_k$. But then the set of r-rowed real square matrices u_i forms a basis of the algebra (u_i) with the same multiplication law as the Cayley non-associative algebra. This is evidently impossible, and the proof of Theorem 5 is complete.

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THE DIRICHLET-NEUMANN PROBLEM FOR THE SPHERE

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1. Introduction. Let S be a sphere with center C, and let D be the domain of points interior to S. Also let B and N be two single-valued continuous functions of position on S. This paper is concerned principally with finding necessary and sufficient conditions that a single-valued function U exist which is harmonic in $D - C$, is continuous in $S + D - C$ and coincides with B on S, and whose normal derivatives coincide with N on S. The problem of showing the existence of such a function U we shall call the Dirichlet-Neumann Problem. C itself may or may not be a removable singular point.

In §7, we use the results of the first part of the paper to prove a uniqueness theorem concerning the Dirichlet-Neumann Problem for regions more general than a sphere. Theorem II on potentials for double layers in §3 is also believed to be new.

2. Necessary Conditions. Let S_1 be a sphere of radius R and center C, and let us choose a system of spherical coordinates (r, ϕ, θ) with C as origin, ϕ and θ being the longitude and colatitude respectively. Let $B(\phi, \theta)$ and $N(\phi, \theta)$ be continuous single-valued functions of ϕ and θ and let $U(r, \phi, \theta)$ be a single-valued function harmonic in the domain $D - C$, where D is the interior of S_1. Our hypotheses are that $U(r, \phi, \theta)$ is continuous in $S_1 + D - C$ and coincides with $B(\phi, \theta)$ on S_1 and that the outer-normal derivatives of U coincide with $N(\phi, \theta)$ on S_1. In this section we shall find conditions that B and N must satisfy if a function U subject to the above hypotheses exists.

Let S_2 be a sphere concentric with S_1 and of radius $r < R$. Then, since S_2 lies in the domain in which U is harmonic, U is continuous and differentiable in ϕ and θ on S_2. U may therefore be expressed in the form

$$U = \sum_{n=0}^{\infty} H_n(\phi, \theta), \tag{1}$$

where $H_n(\phi, \theta)$ is a surface spherical harmonic of degree n. Furthermore this series is uniformly convergent in ϕ and θ on S_2.[1]

Now any surface harmonic $H_n(\phi, \theta)$ is expressible as a linear combination of the $2n + 1$ independent tesseral surface harmonics

$$\begin{cases} C_{nk} = P_n^k(u) \cos k\phi, \\ S_{nk} = P_n^k(u) \sin k\phi \end{cases} \qquad (k = 0, 1, 2, \cdots, n) \tag{2}$$

[1] See O. D. Kellogg, Foundations of Potential Theory, p. 259, Theorem III. Hereafter we shall refer to this book as K.

in the form

$$H_n(\phi, \theta) = \tfrac{1}{2} a_{no} C_{no}(\phi, \theta) + \sum_{k=1}^{n} a_{nk} C_{nk}(\phi, \theta) + \sum_{k=1}^{n} b_{nk} S_{nk}(\phi, \theta), \tag{3}$$

where the a_{nk} and b_{nk} are parameters independent of ϕ and θ and $u = \cos\theta$. The $P_n^k(u)$ are the associated Legendre functions of degree n and order k and are given by the equation

$$P_n^k(u) = (1 - u^2)^{\frac{k}{2}} \frac{d^k}{du^k} P_n(u), \tag{4}$$

where $P_n(u)$ is the ordinary Legendre polynomial of degree n.[2]
$P_n(u)$ is given by the equation[3]

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n} (u^2 - 1)^n. \tag{5}$$

As is well known, the C_{nk} and S_{nk} satisfy the following orthogonality conditions, where $d\omega$ is the element of area on the unit sphere.[4]

$$\begin{cases} \int C_{ni} C_{mj}\, d\omega = \int C_{ni} S_{mj}\, d\omega = \int S_{ni} S_{mj}\, d\omega = 0, \; m \neq n, \\ \int C_{ni} C_{nj}\, d\omega = \int C_{ni} S_{nj}\, d\omega = \int S_{ni} S_{nj}\, d\omega = 0, \; i \neq j, \\ \int C_{ni} S_{ni}\, d\omega = 0, \\ \int C_{ni}^2\, d\omega = \int S_{ni}^2\, d\omega = \dfrac{2\pi}{2n+1} \dfrac{(n+i)!}{(n-i)!}, \; i \neq 0, \\ \int C_{no}^2\, d\omega = \dfrac{4\pi}{2n+1}, \qquad \int S_{no}^2\, d\omega = 0. \end{cases} \tag{6}$$

From (2) we see that S_{no} is identically zero, which accounts for the fact that it does not appear in (3). Also if V_n is a surface spherical harmonic of degree n, $r^n V_n$ and V_n / r^{n+1} are solid spherical harmonics and are harmonic functions throughout all of space except that the latter fails to be harmonic for $r = 0$. Hence $r^n C_{nk}$, $r^n S_{nk}$, C_{nk}/r^{n+1} and S_{nk}/r^{n+1} are harmonic in the closed region bounded by S_1 and S_2.

If we apply Green's second identity to U and $r^n C_{nk}$ for the region bounded by S_1 and S_2 we have

$$\int_{S_2} \left[r^n \frac{\partial U}{\partial n} C_{nk} - U \frac{\partial (r^n C_{nk})}{\partial n} \right]_r dS_2 = \int_1 \left[r^n N C_{nk} - B \frac{\partial (r^n C_{nk})}{\partial n} \right]_R dS_1 \tag{7}$$

[2] K, p. 204.
[3] K, p. 131.
[4] MacMillan, The Theory of the Potential, p. 376.

which reduces to

$$(8)\qquad \begin{cases} r^n \int_{S_2} \dfrac{\partial U}{\partial n} C_{nk}\, dS_2 - nr^{n-1} \int_{S_2} UC_{nk}\, dS_2 = R^n \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad - nR^{n-1} \int_{S_1} UC_{nk}\, dS_1. \end{cases}$$

Similarly applying Green's identity to U and C_{nk}/r^{n+1} for the same region we find

$$(9)\qquad \begin{cases} \dfrac{1}{r^{n+1}} \int_{S_2} \dfrac{\partial U}{\partial n} C_{nk}\, dS_2 + \dfrac{n+1}{r^{n+2}} \int_{S_2} UC_{nk}\, dS_2 = \dfrac{1}{R^{n+1}} \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad + \dfrac{n+1}{R^{n+2}} \int_{S_1} BC_{nk}\, dS_1. \end{cases}$$

If we multiply (9) by r^{2n+1} and then subtract (8) from the result we find

$$(10)\qquad \begin{cases} (2n+1) r^{n-1} \int_{S_2} UC_{nk}\, dS_2 = \left[\dfrac{r^{2n+1}}{R^{n+1}} - R^n\right] \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad + \left[\dfrac{(n+1) r^{2n+1}}{R^{n+2}} + nR^{n-1}\right] \int_{S_1} BC_{nk}\, dS_1. \end{cases}$$

Now the series $\sum_{n=0}^{\infty} H_n(\phi, \theta)$ whose sum is U is uniformly convergent on S_2 and hence continues to be so if each term be multiplied by C_{nk}. Because of the uniform convergence we may integrate term-wise and obtain

$$(11)\qquad \int_{S_2} UC_{nk}\, dS_2 = \sum_{i=0}^{\infty} \int_{S_2} C_{nk} H_i(\phi, \theta)\, dS_2.$$

However, the integral of the product of two surface harmonics of different degrees over a sphere is zero. Hence the sum on the right of (11), where $dS_2 = r^2 d\omega$, reduces to

$$r^2 \int C_{nk} H_n(\phi, \theta)\, d\omega.$$

Substituting for H_n the expression given in (3) and using the relations in (6) we have finally for (11)

$$(12)\qquad \int_{S_2} UC_{nk}\, dS_2 = \frac{2\pi}{2n+1} \frac{(n+k)!}{(n-k)!} r^2 a_{nk}.$$

If we now substitute for the integral on the left of (10) the expression given by (12) and solve for a_{nk}, setting $dS_1 = R^2 d\omega$, we obtain

$$(13)\qquad a_{nk} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!} R\left[\frac{r^n}{R^n} - \frac{R^{n+1}}{r^{n+1}}\right]\int N(\phi,\theta) C_{nk}\, d\omega + \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\left[\frac{(n+1)r^n}{R^n} + n\frac{R^{n+1}}{r^{n+1}}\right]\int B(\phi,\theta) C_{nk}\, d\omega.$$

By applying Green's identity to U and $r^n S_{nk}$ and then to U and S_{nk}/r^{n+1} for the region bounded by S_1 and S_2 we find by a similar procedure,

$$(14)\qquad b_{nk} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!} R\left[\frac{r^n}{R^n} - \frac{R^{n+1}}{r^{n+1}}\right]\int N(\phi,\theta) S_{nk}\, d\omega + \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\left[\frac{(n+1)r^n}{R^n} + n\frac{R^{n+1}}{r^{n+1}}\right]\int B(\phi,\theta) S_{nk}\, d\omega.$$

Let the variables of integration ϕ and θ in (13) and (14) be changed to ϕ' and θ' to distinguish them from the ϕ and θ which are the arguments of H_n. Also let $\cos\theta' = u'$. Now the arc α of the great circle on the unit sphere joining the points (ϕ, θ) and (ϕ', θ') is given by the equation

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Furthermore[5]

$$(15)\qquad P_n(\cos\alpha) = P_n(u)P_n(u') + 2\sum_{k=1}^{\infty}\frac{(n-k)!}{(n+k)!}\{C_{nk}(\phi,\theta)C_{nk}(\phi',\theta') + S_{nk}(\phi,\theta)S_{nk}(\phi',\theta')\}.$$

If now we substitute the values of a_{nk} and b_{nk} given by (13) and (14) in (3) and make use of the relation (15) we find

$$(16)\qquad H_n(\phi,\theta) = \frac{1}{4\pi}\frac{r^n}{R^n}\int [(n+1)B + RN]P_n(\cos\alpha)\, d\omega + \frac{1}{4\pi}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN]P_n(\cos\alpha)\, d\omega.$$

Let us write

$$(17)\qquad H_n(\phi,\theta) = H_n^{(1)}(\phi,\theta) + H_n^{(2)}(\phi,\theta),$$

where

$$(18)\qquad H_n^{(1)}(\phi,\theta) = \frac{1}{4\pi}\frac{r^n}{R^n}\int [(n+1)B + RN]P_n(\cos\alpha)\, d\omega$$

and

$$(19)\qquad H_n^{(2)}(\phi,\theta) = \frac{1}{4\pi}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN]P_n(\cos\alpha)\, d\omega.$$

[5] MacMillan, l.c., p. 380.

which reduces to

$$(8)\qquad \begin{cases} r^n \int_{S_2} \frac{\partial U}{\partial n} C_{nk}\, dS_2 - nr^{n-1} \int_{S_2} UC_{nk}\, dS_2 = R^n \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad - nR^{n-1} \int_{S_1} UC_{nk}\, dS_1. \end{cases}$$

Similarly applying Green's identity to U and C_{nk}/r^{n+1} for the same region we find

$$(9)\qquad \begin{cases} \frac{1}{r^{n+1}} \int_{S_2} \frac{\partial U}{\partial n} C_{nk}\, dS_2 + \frac{n+1}{r^{n+2}} \int_{S_2} UC_{nk}\, dS_2 = \frac{1}{R^{n+1}} \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad + \frac{n+1}{R^{n+2}} \int_{S_1} BC_{nk}\, dS_1. \end{cases}$$

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$$(10)\qquad \begin{cases} (2n+1) r^{n-1} \int_{S_2} UC_{nk}\, dS_2 = \left[\frac{r^{2n+1}}{R^{n+1}} - R^n \right] \int_{S_1} NC_{nk}\, dS_1 \\ \qquad\qquad + \left[\frac{(n+1) r^{2n+1}}{R^{n+2}} + nR^{n-1} \right] \int_{S_1} BC_{nk}\, dS_1. \end{cases}$$

Now the series $\sum_{n=0}^{\infty} H_n(\phi, \theta)$ whose sum is U is uniformly convergent on S_2 and hence continues to be so if each term be multiplied by C_{nk}. Because of the uniform convergence we may integrate term-wise and obtain

$$(11)\qquad \int_{S_2} UC_{nk}\, dS_2 = \sum_{i=0}^{\infty} \int_{S_2} C_{nk} H_i(\phi, \theta)\, dS_2.$$

However, the integral of the product of two surface harmonics of different degrees over a sphere is zero. Hence the sum on the right of (11), where $dS_2 = r^2 d\omega$, reduces to

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Substituting for H_n the expression given in (3) and using the relations in (6) we have finally for (11)

$$(12)\qquad \int_{S_2} UC_{nk}\, dS_2 = \frac{2\pi}{2n+1} \frac{(n+k)!}{(n-k)!} r^2 a_{nk}.$$

If we now substitute for the integral on the left of (10) the expression given by (12) and solve for a_{nk}, setting $dS_1 = R^2 d\omega$, we obtain

$$(13)\qquad a_{nk} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!} R\left[\frac{r^n}{R^n} - \frac{R^{n+1}}{r^{n+1}}\right]\int N(\phi,\theta) C_{nk}\, d\omega + \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\left[\frac{(n+1)r^n}{R^n} + n\frac{R^{n+1}}{r^{n+1}}\right]\int B(\phi,\theta) C_{nk}\, d\omega.$$

By applying Green's identity to U and $r^n S_{nk}$ and then to U and S_{nk}/r^{n+1} for the region bounded by S_1 and S_2 we find by a similar procedure,

$$(14)\qquad b_{nk} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!} R\left[\frac{r^n}{R^n} - \frac{R^{n+1}}{r^{n+1}}\right]\int N(\phi,\theta) S_{nk}\, d\omega + \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\left[\frac{(n+1)r^n}{R^n} + n\frac{R^{n+1}}{r^{n+1}}\right]\int B(\phi,\theta) S_{nk}\, d\omega.$$

Let the variables of integration ϕ and θ in (13) and (14) be changed to ϕ' and θ' to distinguish them from the ϕ and θ which are the arguments of H_n. Also let $\cos\theta' = u'$. Now the arc α of the great circle on the unit sphere joining the points (ϕ, θ) and (ϕ', θ') is given by the equation

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Furthermore[5]

$$(15)\qquad P_n(\cos\alpha) = P_n(u)P_n(u') + 2\sum_{k=1}^{\infty}\frac{(n-k)!}{(n+k)!}\{C_{nk}(\phi,\theta)C_{nk}(\phi',\theta') + S_{nk}(\phi,\theta)S_{nk}(\phi',\theta')\}.$$

If now we substitute the values of a_{nk} and b_{nk} given by (13) and (14) in (3) and make use of the relation (15) we find

$$(16)\qquad H_n(\phi,\theta) = \frac{1}{4\pi}\frac{r^n}{R^n}\int [(n+1)B + RN] P_n(\cos\alpha)\, d\omega + \frac{1}{4\pi}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN] P_n(\cos\alpha)\, d\omega.$$

Let us write

$$(17)\qquad H_n(\phi,\theta) = H_n^{(1)}(\phi,\theta) + H_n^{(2)}(\phi,\theta),$$

where

$$(18)\qquad H_n^{(1)}(\phi,\theta) = \frac{1}{4\pi}\frac{r^n}{R^n}\int [(n+1)B + RN] P_n(\cos\alpha)\, d\omega$$

and

$$(19)\qquad H_n^{(2)}(\phi,\theta) = \frac{1}{4\pi}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN] P_n(\cos\alpha)\, d\omega.$$

[5] MacMillan, l.c., p. 380.

By hypothesis, $B(\phi, \theta)$ and $N(\phi, \theta)$ are bounded, and it is well known that $|P_n(\cos\alpha)| \leq 1$. Furthermore, since $r < R$,

$$\lim_{n\to\infty} \frac{r^n}{R^n}(n+1) = 0.$$

Hence it follows that $H_n^{(1)}(\phi, \theta)$ approaches zero as n approaches infinity for all $r < R$ and therefore that the power series in r/R, namely,

$$\sum_{n=0}^{\infty} H_n^{(1)}(\phi, \theta) \tag{20}$$

is convergent for all $r < R$. Furthermore, it is clear that the convergence is uniform with respect to ϕ and θ on any sphere of radius r with center C. Since $\sum_{n=0}^{\infty} H_n(\phi, \theta)$ is known to be convergent it follows that

$$\sum_{n=0}^{\infty} H_n^{(2)}(\phi, \theta) \tag{21}$$

is also convergent and hence that

$$\lim_{n\to\infty} H_n^{(2)}(\phi, \theta) = 0$$

for $0 < r < R$.

Now let a_{nk} and b_{nk} be written in the form

$$a_{nk} = a_{nk}^{(1)} + a_{nk}^{(2)}, \tag{22}$$

$$b_{nk} = b_{nk}^{(1)} + b_{nk}^{(2)}, \tag{23}$$

where

$$a_{nk}^{(2)} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN]C_{nk}\, d\omega, \tag{24}$$

$$b_{nk}^{(2)} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\frac{R^{n+1}}{r^{n+1}}\int [nB - RN]\, S_{nk}\, d\omega. \tag{25}$$

Since we shall not need $a_{nk}^{(1)}$ and $b_{nk}^{(1)}$ in the following we shall not write them explicitly. If each side of (19) be multiplied by $\cos k\phi$ and then integrated from 0 to 2π we have

$$\int_0^{2\pi} H_n^{(2)}(\phi, \theta)\cos k\phi\, d\phi = \tag{26}$$

$$\frac{1}{4\pi}\int\int_0^{2\pi} [nB - RN]P_n(\cos\alpha)\cos k\phi\, d\phi\, d\omega,$$

where the interchange in the order of integration on the right is evidently allowable. Now referring to (15) and (2) and making use of the well-known relations

$$\int_0^{2\pi} \sin m\phi \cos k\phi \, d\phi = \int_0^{2\pi} \cos m\phi \cos k\phi \, d\phi = 0, m \neq k,$$

$$\int_0^{2\pi} \cos^2 k\phi \, d\phi = \begin{cases} \pi, k \neq 0 \\ 2\pi, k = 0, \end{cases}$$

we see that

$$\int_0^{2\pi} P_n(\cos \alpha) \cos k\phi \, d\phi = \tag{27}$$

$$2\pi \frac{(n-k)!}{(n+k)!} P_n^k(\cos\theta) C_{nk}(\phi', \theta') \quad (k = 0, 1, \ldots, n).$$

Hence (26) becomes

$$\int_0^{2\pi} H_n^{(2)}(\phi, \theta) \cos k\phi \, d\phi \tag{28}$$

$$= \frac{1}{2} \frac{(n-k)!}{(n+k)!} \frac{R^{n+1}}{r^{n+1}} P_n^k(\cos\theta) \int [nB - RN] C_{nk}(\phi', \theta') \, d\omega.$$

Now since $\lim_{n \to \infty} H_n^{(2)}(\phi, \theta) = 0$ it follows that the left side of (28) approaches zero as n approaches infinity, and hence that the limit of the right side must be zero. Furthermore, since $H_n^{(2)}$ is independent of k the expression on the right must converge to zero uniformly with respect to k. Let us write

$$I_c(n, k) = \int [nB - RN] C_{nk}(\phi', \theta') \, d\omega \tag{29}$$

and

$$\lambda = \frac{R}{r}. \tag{30}$$

We may then express the above result as follows.

A necessary condition that our boundary value problem have a solution is

$$\lim_{n \to \infty} \frac{(n-k)!}{(n+k)!} \lambda^n P_n^k(\cos\theta) \, I_c(n, k) = 0, \tag{31}$$

where the limit is approached uniformly as to k. If we set

$$I_s(n, k) = \int [nB - RN] \, S_{nk}(\phi', \theta') \, d\omega \tag{32}$$

we obtain by similar arguments the condition

$$\lim_{n \to \infty} \frac{(n-k)!}{(n+k)!} \lambda^n P_n^k(\cos\theta) I_s(n, k) = 0 \tag{33}$$

uniformly as to k.

From these conditions, (31) and (33), we proceed to derive others which are independent of the variable θ, which enters in them through the factor $P_n^k(\cos\theta)$.

From the definitions of C_{nk} given in (2) and from the two relations in (6) involving the square of this function, we obtain, upon carrying out the integration with respect to ϕ, the well-known formula

$$\int_{-1}^{1} [P_n^k(u)]^2\,du = \frac{2}{2n+1}\frac{(n+k)!}{(n-k)!}\quad (k = 0, 1, \ldots, n). \tag{34}$$

From this it follows that for any value of n or k a value u_1 of u can be found for which

$$P_n^k(u_1) \geq \left[\frac{2}{2n+1}\frac{(n+k)!}{(n-k)!}\right]^{\frac{1}{2}}$$

or

$$\left[(2n+1)\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}} P_n^k(u_1) > 1. \tag{35}$$

Since (31) may be written

$$\lim_{n\to\infty}\left[\frac{1}{2n+1}\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\left[(2n+1)\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}} P_n^k I_c(n,k) = 0, \tag{36}$$

and since (33) may be similarly written, it follows by (35) that

$$\lim_{n\to\infty}\left[\frac{1}{2n+1}\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n I_c(n,k) = 0 \tag{37}$$

and

$$\lim_{n\to\infty}\left[\frac{1}{2n+1}\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n I_s(n,k) = 0 \tag{38}$$

uniformly as to k.

The factor $1/(2n+1)^{\frac{1}{2}}$ in the last two expressions may be removed as a consequence of the following lemma.

LEMMA. *If $F(n)$ be a function of n such that $\lim_{n\to\infty}\lambda^n F(n) = 0$, where λ is any arbitrary constant greater than unity, and if $f(n)$ be a function such that $\lim_{n\to\infty} f(n)/\lambda^n = 0$, then*

$$\lim_{n\to\infty}\lambda^n F(n) f(n) = 0.$$

For if u be a constant greater than λ we may write

$$\lambda^n F(n) f(n) \equiv [u^n F(n)]\left[\frac{f(n)}{\left(\frac{u}{\lambda}\right)^n}\right]$$

and, since the limit of each of the expressions in the brackets is zero by hypotheses, the lemma follows.

If we take for $f(n)$ of the lemma $(2n + 1)^{\frac{1}{2}}$ we obtain from (37) and (38), substituting for I_c and I_s their values given in (29) and (32), the following necessary conditions;

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int[nB - RN]\,C_{nk}(\phi',\theta')\,d\omega = 0 \tag{39}$$

and

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int[nB - RN]\,S_{nk}(\phi',\theta')\,d\omega = 0 \tag{40}$$

uniformly as to k. As a consequence we may state the following theorem.

THEOREM I. *If U be harmonic in the interior of a sphere S_1 except possibly at the center, necessary conditions that U take on the surface of S_1 the values of the continuous single-valued function $B(\phi, \theta)$ and that the normal derivative of U take on S_1 the value of the continuous single-valued function $N(\phi, \theta)$ are that the relations* (39) *and* (40) *be satisfied, where λ is any arbitrary constant greater than unity and the approach to zero is uniform with respect to k.*

3. **Sufficiency of the Conditions.** Let us now form with the two given functions $B(\phi, \theta)$ and $N(\phi, \theta)$ the spherical harmonic $H_n(\phi, \theta)$ given by equation (3), where the coefficients a_{nk} and b_{nk} are given by (13) and (14), and consider the infinite series

$$\sum_{n=0}^{\infty} H_n(\phi, \theta). \tag{41}$$

We shall show in this and the following two sections that, provided conditions (39) and (40) are satisfied, the series (41) defines a function U which is a solution of the proposed Dirichlet-Neumann problem. The present section will be devoted to showing that (41) is convergent and defines an harmonic function at all interior points of S_1 with the possible exception of the center C.

Just as in §2, we may express H_n in the form

$$H_n(\phi, \theta) = H_n^{(1)}(\phi, \theta) + H_n^{(2)}(\phi, \theta), \tag{42}$$

where

$$H_n^{(2)}(\phi, \theta) = \tfrac{1}{2}\,a_{no}^{(2)}\,C_{no} + \sum_{k=1}^{n} a_{nk}^{(2)}\,C_{nk} + \sum_{k=1}^{n} b_{nk}^{(2)}\,S_{nk}, \tag{43}$$

the $a_{nk}^{(2)}$ and $b_{nk}^{(2)}$ being given by (24) and (25). The $H_n^{(1)}$ is then determinate and will be considered later. We now wish to show that

$$\sum_{n=0}^{\infty} H_n^{(2)} \tag{44}$$

is convergent at every interior point of S_1 other than C.

If we square each side of (43) and then integrate each side over the unit sphere we get, using relations (6),

$$(45) \qquad \int [H_n^{(2)}(\phi, \theta)]^2 \, d\omega = \frac{2\pi}{2n+1} \left\{ \frac{1}{2} [a_{no}^{(2)}]^2 + \sum_{k=1}^{n} \frac{(n+k)!}{(n-k)!} [a_{nk}^{(2)}]^2 + \sum_{k=1}^{n} \frac{(n+k)!}{(n-k)!} [b_{nk}^{(2)}]^2 \right\}.$$

Now it appears by comparison of (24) and (25) with (39) and (40) that the latter two conditions are merely

$$\lim_{n \to \infty} \frac{1}{\lambda} \left[\frac{(n+k)!}{(n-k)!} \right]^{\frac{1}{2}} a_{nk}^{(2)} = 0$$

and

$$\lim_{n \to \infty} \frac{1}{\lambda} \left[\frac{(n+k)!}{(n-k)!} \right]^{\frac{1}{2}} b_{nk}^{(2)} = 0$$

uniformly as to k. Hence it follows from the last three relations that, given an $\epsilon > 0$, we can find a constant K such that

$$(46) \qquad \int [H_n^{(2)}]^2 \, d\omega < 2\pi\epsilon^2 \text{ for } n > K.$$

It is to be noticed that K in this relation depends upon r since each of the a_{nk} and b_{nk} depend upon r through the factor $(R/r)^n$ which occurs in them. However, it is clear that if for a given ϵ and r the K be determined, relation (46) will hold for the same ϵ and all greater values of r.

Now let S_2 be any sphere of radius r_2 and concentric with S_1. Let S_3 and S_4, of radii r_3 and r_4, be two other spheres concentric with S_1 and such that

$$0 < r_3 < r_2 < r_4.$$

Then, given an ϵ, we can determine a K such that (46) holds for $r = r_3$ and hence also, because of the remarks in the last paragraph, for all values of r satisfying the relation

$$r_4 \geq r \geq r_3.$$

Hence if we integrate $[H_n^{(2)}]^2$ throughout the volume V_{34} bounded by S_3 and S_4 we find

$$(47) \qquad \int_{V_{34}} [H_n^{(2)}]^2 \, dV < 2\pi\epsilon(r_4^2 - r_3^2).$$

From this it follows that $H_n^{(2)}$ *converges in the mean*[6] to zero in the closed region V_{34}. Furthermore it is clear from the definition of $H_n^{(2)}$ that it is an harmonic function in V_{34}. It therefore follows that in any closed region con-

[6] For a brief discussion of convergence in the mean, see K, p. 267.

tained in V_{34} the function $H_n^{(2)}$ converges uniformly to a harmonic limiting function.[7] In particular, $H_n^{(2)}(\phi, \theta)$ converges uniformly to a continuous limit $L(\phi, \theta)$ on S_2 and we may write

$$\lim_{n\to\infty} \int_{S_2} [H_n^{(2)}]^2 \, dS = \int_{S_2} L^2 \, dS. \tag{48}$$

But from (46) it follows that the above limit is zero and hence that $L = 0$. Since S_2 is any sphere concentric with S_1 we have proved that

$$\lim_{n\to\infty} H_n^{(2)} = 0 \tag{49}$$

at all points of space other than C and furthermore that this limit is approached uniformly at all points lying on a sphere with center C.

Now by (19)

$$\sum_{n=0}^{\infty} H_n^{(2)} = \sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{R}{r}\right)^{n+1} \int [nB - RN] \, P_n(\cos\alpha) \, d\omega. \tag{50}$$

Since (49) holds for all values of $r > 0$ and (50) is a *power* series in R/r it follows that this series is convergent for all $r > 0$. Furthermore, since (49) holds uniformly in ϕ and θ for a fixed r, (50) is uniformly convergent on the sphere of radius r with center at C.

Since in §2 we have shown that the series

$$\sum_{n=0}^{\infty} H_n^{(1)} = \sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{r}{R}\right)^{n} \int [(n+1) \, B + RN] \, P_n(\cos\alpha) \, d\omega \tag{51}$$

is convergent for all values of $r < R$, and uniformly so with respect to ϕ and θ on the sphere of radius r and center C, it follows that the series

$$\sum_{n=0}^{\infty} H_n = \sum_{n=0}^{\infty} H_n^{(1)} + \sum_{n=0}^{\infty} H_n^{(2)} \tag{52}$$

is convergent for all values of r satisfying the relation

$$0 < r < R$$

and that the convergence is uniform with respect to ϕ and θ on the sphere of radius r and center C. The series (51) thus defines a function at all interior points of S_1 other than C. Let us designate this function by

$$U(r, \phi, \theta) = \sum_{n=0}^{\infty} H_n. \tag{53}$$

[7] K, p. 268, Theorem X.

We must now show that U is harmonic at all points (r, ϕ, θ) at which it is defined. Let S_2 and S_3 be two spheres of radii r_2 and r_3 respectively with centers at C and such that $r_2 > r > r_3$. Now from the definition of the H_n it follows that $s_n = \sum_{i=0}^{n} H_i$ is an infinite sequence harmonic in the domain D bounded by S_2 and S_3. Furthermore, as follows from the discussion just above, s_n is uniformly convergent on the boundary of D. Hence by Harnack's first theorem on convergence[8] we know that s_n converges to a function which is harmonic in D and hence harmonic at (r, ϕ, θ), as was to be proved.

4. **The Boundary Values of U.** In this section we wish to show that the function U given by the equation

$$U(r, \phi, \theta) = \sum_{n=0}^{\infty} H_n^{(1)} + \sum_{n=0}^{\infty} H_n^{(2)} \tag{54}$$

satisfies the condition

$$\lim_{P \to A} U(P) = B(A), \tag{55}$$

where P is a point interior to S_1 and A a point on S_1.

Let the coordinates of P be (r, ϕ_1, θ_1) and let A' be any point (r', ϕ', θ') for which $r' > r$. If we write $A'P = \rho'$, then

$$\rho' = (r'^2 + r^2 - 2r'r \cos \alpha)^{\frac{1}{2}} = r'(1 - 2\mu \cos \alpha + \mu^2)^{\frac{1}{2}},$$

where $\cos \alpha = \cos \theta_1 \cos \theta' + \sin \theta_1 \sin \theta' \cos (\phi_1 - \phi')$ and $\mu = r/r'$. Then $1/\rho'$ may be expanded in a series of zonal harmonics in the form[9]

$$\frac{1}{\rho'} = \frac{1}{r'} \sum_{n=0}^{\infty} P_n (\cos \alpha) \frac{r^n}{r'^n}. \tag{56}$$

This series is convergent for all $r' > r$ and we may write for $r' = R$

$$\frac{1}{\rho} = \frac{1}{R} \sum_{n=0}^{\infty} P_n (\cos \alpha) \frac{r^n}{R^n}. \tag{57}$$

Now since $| P_n (\cos \alpha) | \leq 1$ series (57) is dominated by the geometric series

$$\frac{1}{R} \sum_{n=0}^{\infty} \frac{r^n}{R^n}$$

and hence is uniformly convergent in all the variables involved. We may therefore multiply each side by $B(\phi', \theta')$ and integrate over the

[8] K, p. 248, Theorem I.

[9] K, p. 135. His r is our ρ'.

surface of the sphere S_1, the variables of integration being ϕ' and θ'. We thus have

$$\sum_{n=0}^{\infty} \frac{r^n}{R^n} \int_{S_1} BP_n(\cos\alpha)\, dS = R \int_{S_1} \frac{B\, dS}{\rho}. \tag{58}$$

Also the Poisson integral

$$\frac{R^2 - r^2}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho^3}\, dS \tag{59}$$

can be expanded in the well-known form[10]

$$\sum_{n=0}^{\infty} (2n+1) \left[\frac{1}{4\pi R^2} \int_{S_1} B(\phi', \theta') P_n(\cos\alpha)\, dS \right] \frac{r^n}{R^n}. \tag{60}$$

Now by (51) we have

$$\begin{aligned} \sum_{n=0}^{\infty} H_n^{(1)} &= \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) \left[\frac{1}{4\pi R^2} \int_{S_1} BP_n(\cos\alpha)\, dS \right] \frac{r^n}{R^n} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4\pi R^2} \left[\int_{S_1} BP_n(\cos\alpha)\, dS \right] \frac{r^n}{R^n} \\ &+ \sum_{n=0}^{\infty} \frac{1}{4\pi R} \left[\int_{S_1} NP_n(\cos\alpha)\, dS \right] \frac{r^n}{R^n} \end{aligned} \tag{61}$$

which, by reference to (58), (59) and (60), becomes

$$\begin{aligned} \sum_{n=0}^{\infty} H_n^{(1)} &= \frac{1}{2} \frac{R^2 - r^2}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho^3}\, dS + \frac{1}{2} \frac{1}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho}\, dS \\ &+ \frac{1}{4\pi} \int_{S_1} \frac{N(\phi', \theta')}{\rho}\, dS. \end{aligned} \tag{62}$$

Now the Poisson integral (59) may also be expressed in the form

$$-\frac{1}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho}\, dS - \frac{1}{2\pi} \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho}\right)}{\partial n}\, dS, \tag{63}$$

where the normal derivative of $1/\rho$ is to be taken at the point (ϕ', θ'). Hence (62) becomes

$$\sum_{n=0}^{\infty} H_n^{(1)} = -\frac{1}{4\pi} \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho}\right)}{\partial n} dS + \frac{1}{4\pi} \int_{S_1} \frac{N(\phi', \theta')}{\rho}\, dS. \tag{64}$$

[10] K, p. 252. His a, ρ, r are our R, r, ρ respectively.

It thus appears that $\sum_{n=0}^{\infty} H_n^{(1)}$ is the sum of the potential of a double layer on S_1 of moment $-B(\phi', \theta')/4\pi$ and of simple layer on S_1 of density $N(\phi', \theta')/4\pi$.

The series (50) cannot be summed in the manner in which (51) was handled since for points interior to S_1 the ratio R/r is greater than unity. However, we may proceed as follows.

Let us first sum the series

$$W_1 = \sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{r}{R}\right)^{n+1} \int [nB - RN] P_n (\cos \alpha)\, d\omega \tag{65}$$

which is simply (50) with R/r replaced by r/R. Now

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{r}{R}\right)^{n+1} \int nBP_n (\cos \alpha)\, d\omega \\ &= \frac{r}{R} \sum_{n=0}^{\infty} \frac{1}{4\pi R^2} \left(\frac{r}{R}\right)^{n} \int_{S_1} nBP_n (\cos \alpha)\, dS \\ &= \frac{r}{R} \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4\pi R^2} \left(\frac{r}{R}\right)^{n} \int_{S_1} (2n+1)\, BP_n (\cos \alpha)\, dS \\ &\quad - \frac{1}{2} \frac{r}{R} \sum_{n=0}^{\infty} \frac{1}{4\pi R^2} \left(\frac{r}{R}\right)^{n} \int_{S_1} BP_n (\cos \alpha)\, dS. \end{aligned} \tag{66}$$

But by (59), (60) and (63) the next to the last series above has the sum

$$\frac{r}{R}\left[-\frac{1}{4\pi} \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho}\right)}{\partial n}\, dS - \frac{1}{8\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho}\, dS \right]. \tag{67}$$

Similarly the sum of the last series above is, by (58),

$$-\frac{r}{R}\left[\frac{1}{8\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho}\, dS \right]. \tag{68}$$

Also by (58) with N replacing B,

$$\begin{aligned} -\sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{r}{R}\right)^{n+1} \int RNP_n (\cos \alpha)\, d\omega &= -\frac{r}{R} \sum_{n=0}^{\infty} \frac{1}{4\pi R^2} \left(\frac{r}{R}\right)^{n} \int_{S_1} RNP_n (\cos \alpha)\, dS \\ &= -\frac{r}{R} \frac{1}{4\pi} \int_{S_1} \frac{N(\phi', \theta')}{\rho}\, dS. \end{aligned} \tag{69}$$

Hence the sum of (65) is

$$\begin{aligned} W_1 = -\frac{r}{R}\Bigg[&\frac{1}{4\pi} \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho}\right)}{\partial n}\, dS + \frac{1}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho}\, dS \\ &+ \frac{1}{4\pi} \int_{S_1} \frac{N(\phi', \theta')}{\rho}\, dS \Bigg]. \end{aligned} \tag{70}$$

Now if P be a point interior to S_1 we wish to find the $\lim_{P \to A} \sum_{n=0}^{\infty} H_n^{(2)}(P)$. Since $\Sigma H_n^{(2)}$ is a continuous function of r, ϕ, θ in any domain D not containing the point C, it is uniformly continuous in D if D is a closed domain. Hence, given an $\epsilon > 0$ we can draw a sphere M of radius $r_\epsilon < R$ about A such that

$$\left| \sum_{n=0}^{\infty} H_n^{(2)}(P) - \sum_{n=0}^{\infty} H_n^{(2)}(Q) \right| < \epsilon, \tag{71}$$

where P and Q are any two points in or on M. Now the transform of M by reciprocal radii with respect to S_1 will be a sphere M' cutting S_1 in the same circle as M. Also the region D common to M and M' is such that it is transformed into itself, the points of D interior to S_1 going into those of D exterior to S_1 and vice versa, while those on S_1 remain invariant. Hence, if we choose a sphere S with center at A and lying in D the transform of any point interior to S will be in M. Hence if in (71) we let $P: (r, \phi, \theta)$ be in S, and interior to S_1, and choose for $Q: (r', \phi, \theta)$ the point which is the transform of P by means of the equation $rr' = R^2$, we shall have

$$\sum_{n=0}^{\infty} H_n^{(2)}(P) = \sum_{n=0}^{\infty} H_n^{(2)}(Q) + \epsilon', \tag{72}$$

where $|\epsilon'| \leq \epsilon$. Now

$$\sum_{n=0}^{\infty} H_n^{(2)}(Q) = \sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{R}{r'}\right)^{n+1} \int [nB - RN] P_n(\cos \alpha)\, d\omega$$

$$= \sum_{n=0}^{\infty} \frac{1}{4\pi} \left(\frac{r}{R}\right)^{n+1} \int [nB - RN] P_n(\cos \alpha)\, d\omega$$

which by (65) is the function W_1. Hence (72) becomes

$$\sum_{n=0}^{\infty} H_n^{(2)}(P) = W_1 + \epsilon'. \tag{73}$$

We thus have from (54), (64), (70) and (73)

$$U(r, \phi, \theta) = U_1 + U_2 + U_3 + \epsilon', \tag{74}$$

where

$$U_1 = -\frac{1}{4\pi} \left(1 + \frac{r}{R}\right) \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho}\right)}{\partial n} dS,$$

$$U_2 = -\frac{r}{4\pi R^2} \int_{S_1} \frac{B(\phi', \theta')}{\rho} dS,$$

$$U_3 = \frac{1}{4\pi} \left(1 - \frac{r}{R}\right) \int_{S_1} \frac{N(\phi', \theta')}{\rho} dS.$$

Now as $P:(r, \phi_1, \theta_1)$ approaches A the double layer potential

$$\int_{S_1} B(\phi', \theta') \frac{\partial\left(\frac{1}{\rho}\right)}{\partial n} dS \tag{75}$$

approaches the limit[11]

$$-2\pi B(\phi, \theta) + \int_{S_1} B(\phi', \theta') \frac{\partial\left(\frac{1}{\rho'}\right)}{\partial n} dS, \tag{76}$$

where in the integral in (76) ρ' is the distance from the point $A:(R, \phi, \theta)$ to $A':(R, \phi', \theta')$. Furthermore, since $\partial(1/\rho')/\partial n$ is to be taken toward the exterior normal of S_1, we have

$$\frac{\partial\left(\frac{1}{\rho'}\right)}{\partial n} = -\frac{1}{\rho'^2} \cos \gamma, \tag{77}$$

where γ is the angle between the vector AA' and the radius OA'. But from the triangle OAA' we have

$$\cos \gamma = \frac{\rho'}{2R}. \tag{78}$$

Hence from (76), (77) and (78) we have

$$\lim_{P \to A} U_1 = B(\phi, \theta) + \frac{1}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho'} dS. \tag{79}$$

Also, since the potential of a simple layer is continuous throughout space, we have

$$\lim_{P \to A} U_2 = -\frac{1}{4\pi R} \int_{S_1} \frac{B(\phi', \theta')}{\rho'} dS \tag{80}$$

and

$$\lim_{P \to A} U_3 = 0. \tag{81}$$

Hence finally from (74) and the last three equations we have

$$\lim_{P \to A(\phi, \theta)} U(P) = B(\phi, \theta). \tag{82}$$

5. **The Normal Derivatives of U on the Boundary.** In this section we wish to show that

$$\lim_{P \to A} \frac{\partial U(P)}{\partial n} = N(\phi, \theta), \tag{83}$$

[11] K, p. 168, Theorem IX; p. 167, Theorem VIII.

where the normal derivative is taken in the direction of the exterior normal and P approaches A along the radius CA. Evidently here $\partial U/\partial n$ is the same as $\partial U/\partial r$.

Since $\sum_{n=0}^{\infty} H_n^{(1)}$ is given by the function on the right of (64), we are justified in writing

$$(84)\qquad \frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(1)} = \frac{\partial}{\partial r}\left[-\frac{1}{4\pi}\int_{S_1} B(\phi',\theta')\,\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS + \frac{1}{4\pi}\int_{S_1}\frac{N(\phi',\theta')}{\rho}\,dS\right].$$

However, we cannot treat (73) in the same way since we know nothing of the behavior of $\partial\epsilon'/\partial r$. We therefore return to (50) and differentiate termwise with respect to r, this being justified since (50) is uniformly convergent for all values of $r \geq r_1 > 0$. Hence

$$(85)\qquad \frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(2)} = -\frac{R^2}{r^2}\sum_{n=0}^{\infty}\frac{1}{4\pi R}\left(\frac{R}{r}\right)^n (n+1)\int [nB - RN]\,P_n(\cos\alpha)\,d\omega.$$

We first consider the function

$$(86)\qquad W_2 = -\frac{r^2}{R^2}\sum_{n=0}^{\infty}\frac{1}{4\pi R}\left(\frac{r}{R}\right)^n (n+1)\int [nB - RN]\,P_n(\cos\alpha)\,d\omega.$$

From (65) we see that

$$(87)\qquad W_2 = -\frac{r^2}{R^2}\frac{\partial W_1}{\partial r}$$

and hence by (70)

$$(88)\qquad W_2 = \frac{r^2}{R^3}\left[\frac{1}{4\pi}\int_{S_1} B(\phi',\theta')\,\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS + \frac{1}{4\pi R}\int_{S_1}\frac{B(\phi',\theta')}{\rho}\,dS + \frac{1}{4\pi}\int_{S_1}\frac{N(\phi',\theta')}{\rho}\,dS\right] + \frac{r^3}{R^3}\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_{S_1} B(\phi',\theta')\,\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS + \frac{1}{4\pi R}\int_{S_1}\frac{B(\phi',\theta')}{\rho}\,dS + \frac{1}{4\pi}\int_{S_1}\frac{N(\phi',\theta')}{\rho}\,dS\right].$$

Now $(\partial/\partial r)\sum_{n=0}^{\infty} H_n^{(2)}$ is a continuous function of r for $r > 0$ and hence if P

and P' be the points (r, ϕ, θ) and (r', ϕ, θ) on the radius OA such that $rr' = R^2$ and $r < r'$ we may write

$$\frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(2)}(P) = \frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(2)}(P') + \epsilon, \tag{89}$$

where $\epsilon \to 0$ as r and r' approach R. But evidently by (85) and (86)

$$\frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(2)}(P') = W_2(P). \tag{90}$$

Hence by (84), (88), (89) and (90) we have

$$\begin{aligned}\frac{\partial U(P)}{\partial r} &= \frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(1)} + \frac{\partial}{\partial r}\sum_{n=0}^{\infty} H_n^{(2)} \\ &= X_1 + X_2 + X_3 + \epsilon,\end{aligned} \tag{91}$$

where

$$\begin{aligned}X_1 = \frac{r^2}{R^3}\left[\frac{1}{4\pi}\int_{S_1} B(\phi', \theta')\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS + \frac{1}{4\pi R}\int_{S_1}\frac{B(\phi', \theta')}{\rho}\,dS\right] \\ + \frac{r^3}{R^3}\frac{\partial}{\partial r}\left[\frac{1}{4\pi R}\int_{S_1}\frac{N(\phi', \theta')}{\rho}\,dS\right],\end{aligned} \tag{92}$$

$$X_2 = \frac{r^2}{4\pi R^3}\int_{S_1}\frac{N(\phi', \theta')}{\rho}\,dS + \left(\frac{r^3}{R^3} + 1\right)\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_{S_1}\frac{N(\phi', \theta')}{\rho}\,dS\right], \tag{93}$$

$$X_3 = -\left(1 - \frac{r^3}{R^3}\right)\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_{S_1} B(\phi', \theta')\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS\right]. \tag{94}$$

Now if we write

$$X_1 = X_1^{(1)} + X_1^{(2)} + X_1^{(3)}, \tag{95}$$

where

$$X_1^{(1)} = \frac{r^2}{R^3}\frac{1}{4\pi}\int_{S_1} B(\phi', \theta')\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS, \tag{96}$$

$$X_1^{(2)} = \frac{r^2}{R^3}\frac{1}{4\pi R}\int_{S_1}\frac{B(\phi', \theta')}{\rho}\,dS, \tag{97}$$

$$X_1^{(3)} = \frac{r^3}{R^3}\frac{\partial}{\partial r}\left[\frac{1}{4\pi R}\int_{S_1}\frac{B(\phi', \theta')}{\rho}\,dS\right], \tag{98}$$

we find by the method by which (75) was treated that

$$\lim_{P \to A} X_1^{(1)} = -\frac{1}{2R} B(\phi, \theta) - \frac{1}{8\pi R^2} \int_{S_1} \frac{B(\phi', \theta')}{\rho'} dS. \tag{99}$$

Also, as is evident,

$$\lim_{P \to A} X_1^{(2)} = \frac{1}{4\pi R^2} \int_{S_1} \frac{B(\phi', \theta')}{\rho'} dS. \tag{100}$$

Now[12]

$$\lim_{P \to A} \frac{\partial}{\partial r} \int_{S_1} \frac{B(\phi', \theta')}{\rho} dS = 2\pi B(\phi, \theta) + \int_{S_1} B(\phi', \theta') \frac{\partial \left(\frac{1}{\rho'}\right)}{\partial n} dS, \tag{101}$$

where the normal derivative is to be taken at the point (ϕ, θ). By the same sort of argument as that following equation (76) we have

$$\frac{\partial \left(\frac{1}{\rho'}\right)}{\partial n} = -\frac{1}{2R\rho'}, \tag{102}$$

although here the derivative is taken at (ϕ, θ) while there at (ϕ', θ'). Hence

$$\lim_{P \to A} X_1^{(3)} = \frac{1}{2R} B(\phi, \theta) - \frac{1}{8\pi R^2} \int_{S_1} \frac{B(\phi', \theta')}{\rho'} dS. \tag{103}$$

Hence from (95), (99), (100) and (103) we have

$$\lim_{P \to A} X_1 = 0. \tag{104}$$

Now let

$$X_2 = X_2^{(1)} + X_2^{(2)}, \tag{105}$$

where

$$X_2^{(1)} = \frac{r^2}{4\pi R^3} \int_{S_1} \frac{N(\phi', \theta')}{\rho} dS \tag{106}$$

and

$$X_2^{(2)} = \left(\frac{r^3}{R^3} + 1\right) \frac{\partial}{\partial r} \left[\frac{1}{4\pi} \int_{S_1} \frac{N(\phi', \theta')}{\rho} dS\right]. \tag{107}$$

Evidently

$$\lim_{P \to A} X_2^{(1)} = \frac{1}{4\pi R} \int_{S_1} \frac{N(\phi', \theta')}{\rho'} dS. \tag{108}$$

[12] K, p. 164, Theorem VI.

By the same method by which (101) was treated we find that

$$\lim_{P \to A} X_2^{(2)} = N(\phi, \theta) - \frac{1}{4\pi R} \int_{S_1} \frac{N(\phi', \theta')}{\rho'} dS. \tag{109}$$

Hence by (105), (108) and (109) we have

$$\lim_{P \to A} X_2 = N(\phi, \theta). \tag{110}$$

If we can show that $\lim_{P \to A} X_3 = 0$, we have obtained our desired result. But in (94), although the factor $(1 - r^3/R^3)$ approaches zero as $r \to R$ we cannot infer that $X_3 \to 0$, since in general, for merely continuous moments, the limit of the normal derivative of the potential of a double layer does not exist when the point at which the derivative is evaluated approaches the point of the double layer at which the normal is drawn. In order to gain the desired result we shall prove theorem II stated below. In proving the theorem we shall use the methods and notation of Kellogg's admirable treatment in Chapter VI of his book.

Let S be a regular surface element for which the standard representation $z = f(x, y)$, where the point (x, y) is in a regular region R, is subject to the further condition that $f(x, y)$ shall have continuous second partial derivatives in R. We now state

THEOREM II. *Let U be the potential of a double distribution on S with a piecewise continuous moment μ, and let p be an interior point of S at which μ is continuous. Also let P be a point on the normal to S at p. Then*

$$\lim_{P \to p} \overline{Pp} \frac{\partial U(P)}{\partial n} = 0,$$

where $\partial U(P)/\partial n$ is the derivative of U evaluated at P in the direction P to p.

The potential U may be expressed in the form

$$U(x, y, z) = \int_s \mu \frac{\partial}{\partial \nu} \frac{1}{r} dS = \int_{s'} \mu \sec \gamma \frac{\partial}{\partial \nu} \frac{1}{r} dS', \tag{111}$$

where (x, y, z) is the point at which U is evaluated, r the distance from (x, y, z) to the point $q(\xi, \eta, \zeta)$ on the surface S at which the normal derivative $(\partial/\partial\nu)(1/r)$ is evaluated, S' the projection of S on the XY plane, and γ the angle between the Z-axis and the normal at q. Now it is possible to find a neighborhood of p of radius c such that for a set of ξ, η, ζ axes, where the ξ, η plane is tangent to S at p and the ζ axis is the normal at p, the portion of S within the sphere of radius c has a standard representation of the form[13] $\zeta = \phi(\xi, \eta)$ such that the function ϕ will have continuous partial derivatives of the second order. With this tangent-normal system of axes the point P of the theorem is on the positive ζ-axis and we shall represent it by the coördinates $(0, 0, z)$. Then

[13] K, pp. 157, 158.

$r^2 = \xi^2 + \eta^2 + (\zeta - z)^2$. Now the potential of the distribution on the portion of S exterior to the above sphere of radius c is analytic at p and hence the limit appearing in the theorem will be zero for this portion. We may therefore neglect this part of S and assume that our entire surface element is given by the equation $\zeta = \phi(\xi, \eta)$.

Now from (111) we have

$$\frac{\partial U}{\partial n} = -\frac{\partial U}{\partial z} = -\int_{s'} \mu \frac{\partial}{\partial z}\left[\frac{\partial}{\partial \nu}\frac{1}{r}\right]_s \sec\gamma \, dS', \tag{112}$$

and if α, β, γ be the direction cosines of the normal to S at q,

$$\frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) = \frac{\partial}{\partial \xi}\left(\frac{1}{r}\right)\cos\alpha + \frac{\partial}{\partial \eta}\left(\frac{1}{r}\right)\cos\beta + \frac{\partial}{\partial \zeta}\left(\frac{1}{r}\right)\cos\gamma$$

$$= \frac{\xi\phi_\xi + \eta\phi_\eta + (z-\zeta)}{r^3}\cos\gamma.$$

If this be differentiated with respect to z, (112) becomes

$$I = \frac{\partial U}{\partial n} = \int_{s'} \mu\left[-\frac{1}{r^3} + \frac{3z^2 - 3(2\zeta - \xi\phi_\xi - \eta\varphi_\eta)\, z + 3\zeta(\zeta - \xi\phi_\xi - \eta\phi_\eta)}{r^5}\right] dS'. \tag{113}$$

Our object is to prove that $\lim_{z\to 0} zI = 0$. We first notice that the moment of the double distribution may be reduced to zero at p by subtracting the potential U_p of a double distribution on S of constant moment μ_p, the value of μ at p. This will not affect the limit of zI, for the potential U_p may be looked upon as a many-valued analytic function in all of space, except at the boundary points of S, whose branches differ by constants.[14] The derivatives of U_p are thus continuous and hence

$$\lim_{z\to 0} z\frac{\partial U_p}{\partial n} = 0.$$

Hence our object will be attained if we show that $\lim_{z\to 0} zI = 0$ on the assumption that μ is zero at p. Also, if the integral I is extended over a region $S' - \sigma'$, where σ' is a small circle about p, zI taken over this region will approach zero with z. Hence if I', an integral with the same integrand as I but extended over a circle σ' with radius a is such that zI' approaches zero with a, uniformly with z, it will follow that $\lim_{z\to 0} zI = 0$. We shall now show that this condition is satisfied. If we write

$$I' = I_1 + 3I_2 + 3I_3 + 3I_4,$$

where

$$I_1 = -\int_{\sigma'} \mu\frac{1}{r^3}\, dS', \quad I_2 = \int_{\sigma'} \mu\frac{z^2}{r^5}\, dS',$$

[14] K, p. 69.

$$I_3 = -\int_{\sigma'} \mu \frac{(2\zeta - \xi\phi_\xi - \eta\phi_\eta)z}{r^5}\, dS',$$

$$I_4 = \int_{\sigma'} \mu \frac{\zeta(\zeta - \xi\phi_\xi - \eta\phi_\eta)}{r^5}\, dS',$$

our purpose will be accomplished if we show that

$$\lim_{a\to 0} zI_i = 0 \qquad (i = 1, 2, 3, 4)$$

uniformly as to z.

Now if the function $\zeta = \phi(\xi, \eta)$ be expanded in a Taylor series, with remainder, about p the origin, the linear terms will vanish since the ξ, η plane is tangent to S at p and we have

$$\zeta = \tfrac{1}{2}[\bar{\phi}_{\xi\xi}\xi^2 + 2\bar{\phi}_{\xi\eta}\xi\eta + \bar{\phi}_{\eta\eta}\eta^2],$$

where the bars on the second derivatives indicate mean values. If M be the bound of absolute values of these derivatives in R, we then have

$$|\zeta| \leq M(\xi^2 + \eta^2). \tag{114}$$

Let us write

$$r'^2 = \xi^2 + \eta^2,\ \rho^2 = r'^2 + z^2, \tag{115}$$

where ρ is the distance from p to the projection (ξ, η) of q, and r' is the distance of $(\xi, \eta, 0)$ from p.

Now

$$\begin{aligned} r^2 &= \xi^2 + \eta^2 + (\zeta - z)^2 \\ &= r'^2 + \zeta^2 - 2\zeta z + z^2 \\ &= \rho^2 + \zeta(\zeta - 2z). \end{aligned}$$

From this we have at once

$$\frac{r^2}{\rho^2} - 1 = \frac{(\zeta - 2z)\zeta}{\rho^2},$$

or

$$\frac{r}{\rho} - 1 = \frac{(\zeta - 2z)\zeta}{\rho^2} \cdot \frac{\rho}{\rho + r}.$$

From this and (114), (115) it follows that

$$\left|\frac{r}{\rho} - 1\right| \leq \frac{r'^2}{r'^2 + z^2} M \left|2z - \zeta\right|. \tag{116}$$

Now if $-\delta \leq z \leq \delta$, δ and a can be chosen so small that, uniformly as to z, the expression on the right of (116) can be made less than any positive quantity, say $1/2$. Then $r > \rho/2$. Now by the law of the mean for integrals,

$$zI_1 = -\int_{\sigma'} \mu \frac{z}{r^3}\, dS' = -\bar{\mu} \int_{\sigma'} \frac{z}{r^3}\, dS'$$

and hence, since $r > \rho/2$,

$$\left|zI_1\right| < \left|8\bar{\mu}\int_{\sigma'}\frac{z}{\rho^3}\,dS'\right|$$

$$= \left|16\pi\bar{\mu}\int^{a}\frac{zr'}{(r'^2+z^2)^{\frac{3}{2}}}\,dr'\right|.$$

If the substitution $r' = z\tan\lambda$ be made, it will be found that the above integral is not greater than unity for any z. Hence, since $\bar{\mu}$ approaches zero with a, it follows that zI_1 also approaches zero uniformly as to z.

The other three integrals may be treated similarly. In I_3 and I_4 we need merely apply the mean value theorem for functions of two variables to $2\zeta - \xi\phi_\xi - \eta\phi_\eta$ and $\zeta - \xi\phi_\xi - \eta\phi_\eta$, remembering that the second derivatives $\phi_{\xi\xi}, \phi_{\xi\eta}, \phi_{\eta\eta}$ are bounded. We find then that $|zI_i|$ $(i = 2, 3, 4)$ are each equal to $\bar{\mu}$ multiplied by a bounded quantity. It thus follows that

$$\lim_{z\to 0} zI = 0$$

and our theorem is proved.

We now return to the consideration of equation (94). It may be written

$$X_3 = -\left(\frac{R^2 + R_r + r^2}{R^3}\right)(R-r)\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_{s_1} B(\phi',\theta')\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS\right]. \tag{117}$$

Now $(R - r)$ is the distance $\overline{PA}$ and hence by the theorem just proved we have

$$\lim_{r\to R}(R-r)\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_{s_1} B(\phi',\theta')\frac{\partial\left(\frac{1}{\rho}\right)}{\partial n}\,dS\right] = 0.$$

Since the remaining factor of (117) approaches $-3/R$ it follows that

$$\lim_{P\to A} X_3 = 0. \tag{118}$$

Hence we have, from (91), (104), (110) and (118),

$$\lim_{P\to A(\phi,\theta)}\frac{\partial U}{\partial r} = N(\phi,\theta).$$

From this equation and (82) it follows that conditions (39) and (40) are sufficient for the existence of the proposed boundary value problem. We may now state the following theorem.

THEOREM III. *Given B and N, two continuous single-valued functions of position on a sphere S of radius R, necessary and sufficient conditions that a function U exist which is harmonic in the interior D of S, except possibly at the center C, which is continuous in $S + D - C$ and coincides with B on S, and is such that the normal derivative $\partial U/\partial n$ approaches N on S, are that the condi-*

tions (39) *and* (40) *be satisfied, where* λ *is any arbitrary constant greater than unity and the approach to zero is uniform with respect to* k. *This solution is unique and is given explicitly by means of equations* (1), (3), (13) *and* (14).

6. A Special Case. If instead of (39) and (40) the stronger conditions

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int nBC_{nk}\,d\omega = 0, \tag{119}$$

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int nBS_{nk}\,d\omega = 0, \tag{120}$$

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int RNC_{nk}\,d\omega = 0, \tag{121}$$

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\lambda^n\int RNS_{nk}\,d\omega = 0 \tag{122}$$

hold[15] uniformly as to k, the series

$$F_1(\phi,\theta) = \sum_{n=0}^{\infty}\frac{1}{4\pi}\left[\frac{(n+1)\,r^n}{R^n} + \frac{nR^{n+1}}{r^{n+1}}\right]\int BP_n(\cos\alpha)\,d\omega \tag{123}$$

and

$$F_2(\phi,\theta) = \sum_{n=0}^{\infty}\frac{R}{4\pi}\left[\frac{r^n}{R^n} - \frac{R^{n+1}}{r^{n+1}}\right]\int NP_n(\cos\alpha)\,d\omega \tag{124}$$

will each be convergent and, by reference to (16), we see that

$$U = \sum_{n=0}^{\infty} H_n(\phi,\theta) = F_1 + F_2.$$

Conditions (119) and (120) result from (39) and (40) if N be taken zero, and hence we have, by the theory of the previous sections, that

$$\lim_{P\to A\,(\phi,\theta)} F_1 = B(\phi,\theta),\ \lim_{P\to A}\frac{\partial F_1}{\partial r} = 0. \tag{125}$$

Similarly (121) and (122) are, except for sign, the special case of (39) and (40) for $B \equiv 0$. Hence

$$\lim_{P\to A} F_2 = 0,\ \lim_{P\to A}\frac{\partial F_2}{\partial r} = N(\phi,\theta). \tag{126}$$

We may therefore state the following theorem.

[15] The factor R in the integrand of (121) and (122) may, of course, be dropped.

THEOREM IV. *Conditions* (119)–(122) *are necessary and sufficient for the solution of the Dirichlet-Neumann problem in the form*

$$U = F_1 + F_2, \tag{127}$$

where F_1 takes the required boundary values B and F_2 the boundary values zero, while the normal derivative of F_1 takes the value zero on the boundary and the normal derivative of F_2 takes the assigned values N. The functions F_1 and F_2 are given uniquely by the equations (123) *and* (124).

By the lemma of §2, it follows that the conditions (119) and (120) are equivalent to the conditions obtained from them by removing the factors n from the integrand. From this and the remark of the last footnote it follows that B and N each satisfy the same two conditions, namely,

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\int XC_{nk}\,d\omega = 0 \tag{128}$$

and

$$\lim_{n\to\infty}\left[\frac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}}\int XS_{nk}\,d\omega = 0 \tag{129}$$

uniformly as to k. We thus have

THEOREM V. *If, when substituted for X, B and N each satisfy the two equations* (128) *and* (129), *the solution of the Dirichlet-Neumann problem with boundary conditions B and N exists and is given by* (127), *where F_1 and F_2 have the properties stated in theorem IV.*

THEOREM VI. *If the solution of the Dirichlet-Neumann problem for boundary conditions B and N exists in the form* (127), *where F_1 and F_2 have the properties stated in theorem IV, then B and N each satisfies the two conditions* (128) *and* (129).

7. **A Uniqueness Theorem.** Let T be a regular region of space[16] and S its boundary. Let P be an interior point of T. Also let $B(x, y, z)$ and $N(x, y, z)$ be two single-valued continuous functions of position on S and let $U(x, y, z)$ be a single-valued function which is harmonic at each point of the domain $T - S - P$, is continuous in $T - P$, and is such that on S the function U coincides with B and the normal derivative of U coincides with N. P is thus a singular point of U which may or may not be removable. Such a function U we shall call a solution of the Dirichlet-Neumann problem for the region T corresponding to the point P, with boundary conditions B and N.

We wish to prove the following uniqueness theorem.

THEOREM VII. *If a solution of the Dirichlet-Neumann problem for the region T and point P with assigned boundary conditions B and N exist, it is unique.*

Let F_1 and F_2 be two solutions for the region T and point P, and satisfying

[16] K, p. 113.

the same boundary conditions B and N. Our theorem will be proved if we can show that the difference

$$F_1 - F_2 = U \tag{130}$$

is identically zero in the domain $T - S - P$.

Since P is an interior point of T, a sphere S_1 can be drawn having P as center and lying entirely in T. Furthermore, since by hypothesis F_1 and F_2 have no singular point except possibly at P, the function U of (130) is harmonic in the regular region T' whose boundary is made up of S and S_1. Also in the same region the function $r^n C_{nk}$ is harmonic, where r is the distance from P and C_{nk} is given by (2). Hence, applying Green's formula to U and $r^n C_{nk}$ for the region T', we have

$$\int_{SS_1} \left[U \frac{\partial (r^n C_{nk})}{\partial m} - r^n C_{nk} \frac{\partial U}{\partial m} \right] dS = 0, \tag{131}$$

where $\partial/\partial m$ indicates the inner normal with respect to T' and hence for the sphere S_1 is the outer normal derivative. Since F_1 and F_2 take identical values on S and have the same normal derivatives on S, (131) reduces to

$$\int_{S_1} \left[U \frac{\partial (r^n C_{nk})}{\partial m} - r^n C_{nk} \frac{\partial U}{\partial m} \right] dS = 0. \tag{132}$$

Let us designate the values of U and $\partial U/\partial m$ on S_1 by $B_1(\phi, \theta)$ and $N_1(\phi, \theta)$ respectively. Then we find after a slight reduction that (132) becomes

$$n \int B_1(\phi, \theta)\, C_{nk}\, d\omega - R \int N_1(\phi, \theta)\, C_{nk}\, d\omega = 0, \tag{133}$$

where R is the radius of S_1 and $d\omega$ is the element of area on the unit sphere.

If we repeat the above process with the functions U and C_{nk}/r^{n+1} we find similarly that

$$(n+1) \int B_1(\phi, \theta) C_{nk}\, d\omega + R \int N_1(\phi, \theta) C_{nk}\, d\omega = 0. \tag{134}$$

Also, from the pair U and $r^n S_{nk}$, we get

$$n \int B_1(\phi, \theta) S_{nk}\, d\omega - R \int N_1(\phi, \theta) S_{nk}\, d\omega = 0, \tag{135}$$

and from the pair U and S_{nk}/r^{n+1},

$$(n+1) \int B_1(\phi, \theta) S_{nk}\, d\omega + R \int N_1(\phi, \theta) S_{nk}\, d\omega = 0. \tag{136}$$

Now, from the results of section 2, U may be expanded in the interior of S_1 in a series of spherical harmonics in the form given by equation (1), where the $H_n(\phi, \theta)$ are given by (3) and the a_{nk} and b_{nk} are given by (13) and (14) respectively, and where of course B and N of the last equations are to be replaced by B_1 and N_1 respectively. That the boundary values and the values of the normal derivatives in terms of which U is expressed in the interior of S_1

are the same as the B_1 and N_1 appearing in equations (133) to (136) follows from the fact that U is harmonic everywhere in $T - S - P$ and hence it and its derivatives are continuous in passing from the interior to the exterior of S_1.

Now by a rearrangement of terms we have

$$(137)\quad a_{nk} = \frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\frac{R^{n+1}}{r^{n+1}}\left[n\int B_1(\phi,\theta)C_{nk}\,d\omega - R\int N_1(\phi,\theta)C_{nk}\,d\omega\right]$$

$$+\frac{1}{2\pi}\frac{(n-k)!}{(n+k)!}\frac{r^n}{R^n}\left[(n+1)\int B_1(\phi,\theta)C_{nk}\,d\omega + R\int N_1(\phi,\theta)C_{nk}\,d\omega\right].$$

Hence by (133) and (134) it follows that

$$a_{nk} = 0.$$

Similarly from (135) and (136) we have

$$b_{nk} = 0.$$

Hence in the region consisting of the interior of S_1 with P removed, U is identically zero. From this it follows[17] that U is identically zero in the domain $T - S - P$ and our theorem is proved.

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[17] K, p. 259, Theorem IV.

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A JORDAN SPACE CURVE NO ARC OF WHICH CAN FORM PART OF A CONTOUR WHICH BOUNDS A FINITE AREA

BY JESSE DOUGLAS

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1. **Description.** The author, sometimes with the collaboration of P. Franklin, has recently given examples, increasing in their simplicity, of Jordan space curves which bound no surface of finite area.[1] In each case the infinite part of the area was concentrated in the neighborhood of a unique singular point of the bounding curve.

The present paper applies a method of condensation of singularities to construct a Jordan space curve with the property stated in the title. The example to be given is the one with the following parametric equations:

$$x = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos 4^n\varphi,$$

(1) or (J)
$$y = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin 4^n\varphi,$$

$$z = \sin \varphi.$$

The infinite series are evidently uniformly convergent, and each term is a continuous periodic function of period 2π, so that the curve is continuous and closed. That there are no double points is readily seen as follows.

The equation

$$z' = z \quad \text{or} \quad \sin \varphi' = \sin \varphi$$

implies

(2)
$$\varphi' \equiv \varphi \quad \text{or} \quad \varphi' \equiv \pi - \varphi \pmod{2\pi}.$$

The second of these possibilities, used in the equation $x' = x$, leaves, after cancellation, only the terms corresponding to $n = 0$:

$$\cos(\pi - \varphi) = \cos\varphi, \quad -\cos\varphi = \cos\varphi, \quad \cos\varphi = 0;$$

therefore

$$\varphi \equiv \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \pmod{2\pi},$$

[1] J. Douglas and P. Franklin, Proceedings of the National Academy of Sciences, 19 (1933), no. 1, pp. 188–191; J. Douglas, the same, no. 2, pp. 269–271; J. Douglas, the same, no. 4, pp. 448–451.

and, respectively,

$$\varphi' \equiv \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \pmod{2\pi},$$

so that again we have the first relation (2). Thus, no two points of the curve (1), call it now J, can be the same unless their parametric values differ by a period.

The transformation of φ into $\varphi + \frac{p}{4^m} \cdot 2\pi$, where p, m are any positive integers, performed on the equations of J, evidently has the effect of modifying these equations only by the addition of finite trigonometric polynomials. Hence if any singularity is present at the point $\varphi = \alpha$, essentially the same singularity will reproduce itself at the everywhere dense set of points whose parameters differ from α by 2π multiplied by any rational fraction with denominator a power of 4. It is this fact which underlies the condensation of the singularity producing infinite area all along the curve J.

An idea of the form of J may be obtained by first considering by themselves its first two equations, representing the orthogonal projection J' of J on the xy-plane. This is the limit of an epicyclic curve generated as follows: a point P_1 moves at uniform speed around a circle of radius 1 about the origin; about P_1 as center and on a circle of radius $\frac{1}{2}$ moves a point P_2 uniformly in the same sense and with four times the angular speed (frequency) of P_1; about P_2 as center and on a circle of radius $\frac{1}{2^2}$ moves a point P_3 uniformly with four times the angular speed of P_2, etc. The curve J' is the limit of the locus of the point P_n as $n \to \infty$. It has infinitely many double points and loops. The adjunction of the third equation, $z = \sin \varphi$, (equivalent to compounding with the motion of the point describing J' a vertical simple harmonic motion with period equal to that of J'), gets rid of all the double points, and converts the loops of J' into spirals on the Jordan curve J.

In the author's previous examples, the singularity producing infinite area was due to the presence in the curve of a spiral whose coils contracted with a sufficient order of slowness towards the unique singular point. In the present example, such slowly contracting spirals are distributed in an everywhere dense manner along the curve.

2. **Proof.** Many readers will already have recognized in the first of the equations (1) an example given by Hadamard in connection with the Dirichlet problem.[2]

Let boundary values on the unit circumference be defined by the function $f(\varphi)$ with Fourier coefficients a_n, b_n. Then if $H(\rho, \varphi)$ denote the harmonic func-

[2] J. Hadamard, Bulletin de la Société Mathématique de France, **34** (1906), pp. 135–139.

tion in the interior of the unit circle with the boundary values $f(\varphi)$ on the circumference, we have

$$H(\rho, \varphi) = \sum_{n=0}^{\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi),$$

and the value of the Dirichlet functional for H is

$$(3) \qquad D(H) = \int\int \left[\left(\frac{\partial H}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial H}{\partial \theta}\right)^2\right] \rho \, d\rho \, d\theta = \sum_{n=1}^{\infty} n (a_n^2 + b_n^2).$$

$D(H)$ is, moreover, a minimum: if G is any other function (of class C') with the boundary values $f(\varphi)$, then $D(G) > D(H)$.

Hadamard remarked that if we take

$$f(\varphi) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos 4^n \varphi,$$

then, by (3),

$$D(H) = \sum_{n=0}^{\infty} \frac{4^n}{4^n} = +\infty;$$

consequently $D(G) = +\infty$ for all functions G (of class C') with the boundary values $f(\varphi)$.

That the infinite part of $D(H)$ comes from the rim of the unit circle is obvious, since the integral defining $D(H)$ taken over any concentric circle of radius less than 1 has a finite value.

But more is true, as I shall now show; every region of the unit circle which abuts the circumference along an arc, however small, contributes $+\infty$ to $D(H)$.

Let the unit circle be divided into two semi-circles; then at least one of these must contribute $+\infty$ to the value of $D(H)$. Let this semi-circle be divided into two quadrants; then at least one of these must contribute $+\infty$ to $D(H)$. Continuing in this way, by successive bisection, we arrive at the result that there exists a radius, call its polar angle α, such that every sector containing this radius in its interior contributes $+\infty$ to the value of $D(H)$.

Furthermore, it is evidently the part of the sector adjacent to the circumference that contributes $+\infty$ to the integral; that is, the Dirichlet integral over the arbitrarily small region $1 - \epsilon < \rho < 1, \alpha - \delta < \varphi < \alpha + \delta$ is $+\infty$.

The important remark[3] to be made now is that the transformation of φ into $\varphi + \frac{p}{4^m} \cdot 2\pi$, where p and m are any positive integers, changes H only by a finite sum of terms of the form $\rho^n(a_n \cos n\varphi + b_n \sin n\varphi)$ (that is, a finite harmonic

[3] This is essentially the same remark as that used by Hadamard in his thesis to prove that the unit circumference is a natural boundary of the power series (4) below, of which the harmonic function H is the real part. See Hadamard, *La série de Taylor et son prolongement analytique*, Paris, 1901, p. 32. A previous proof had been given by Weierstrass.

polynomial). It follows that every region of the interior of the unit circle which touches the circumference along an arc containing any of the points $\alpha + \frac{p}{4^m} \cdot 2\pi$ contributes $+\infty$ to the value of the Dirichlet integral. Since, however, these points are distributed in an everywhere dense manner over the circumference, we have simply the result that over every interior region touching the circumference along an arc, however small, the value of the Dirichlet integral is $+\infty$.

Consider now the harmonic functions in the unit circle defined by the first two of the equations (1) as boundary values, that is

$$H_1 = \sum_{n=0}^{\infty} \frac{1}{2^n} \rho^{4^n} \cos 4^n\varphi, \quad H_2 = \sum_{n=0}^{\infty} \frac{1}{2^n} \rho^{4^n} \sin 4^n\varphi.$$

They are conjugate harmonic, and define a conformal map of the unit circle on a Riemann surface over the xy-plane, bounded by the curve J' described in the first section of this paper. The function of a complex variable associated with this map is (with $w = x + iy$, $z = \rho e^{i\varphi}$)

$$w = F(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} z^{4^n}, \tag{4}$$

and the area of any portion of the map is

$$\iint |F'(z)|^2 \, d\sigma \quad (d\sigma = \rho \, d\rho \, d\varphi)$$

taken over the corresponding region of the unit circle, which is the same as the integral $D(H_1)$ over the same region.

Consequently, the area of any region of the Riemann surface adjacent to an arc, however small,[4] of its boundary J' is $+\infty$.

Since, as the author has shown,[5] the Plateau problem of minimal surfaces reduces to the Riemann problem of conformal mapping in the case of two dimensions, the Riemann surface defined by the conformal map has the least area of any Riemann surface bounded by J'. We may therefore state the property of the preceding paragraph for any Riemann surface bounded by J'.

If, now, S is any surface bounded by the space curve J, then the orthogonal projection of S on the xy-plane is a Riemann surface bounded by J'. Since orthogonal projection diminishes area, we have finally the result that the area of any region of S which touches J along an arc, however small,[4] is infinite.

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[4] Meaning however small its chord, or the maximum distance between two of its points. Every arc of J' or J has infinite length.

[5] *Solution of the problem of Plateau*, Trans. Amer. Math. Soc., **33** (1931), pp. 263–321; *The problem of Plateau*, Bull. Amer. Math. Soc., **39** (1933), no. 4, pp. 227–251.

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DARSTELLUNG DER SEMI-VEKTOREN ALS GEWÖHNLICHE VEKTOREN VON BESONDEREM DIFFERENTIATIONS CHARAKTER

VON A. EINSTEIN UND W. MAYER

(Received November 8, 1933)

Der Einbau der Semi-Vektoren bzw. Spinoren in die allgemeine Relativitäts-Theorie bzw. in den Riemann'schen Raum geschah bisher in der Weise, dass in jedem Punkte der Mannigfaltigkeit ein orthogonales lokales Koordinatensystem eingeführt wurde, auf welches die Semivektoren bezogen wurden. Dies Lokalsystem seinerseits wurde mit Bezug auf das Gausssche Koordinatensystem durch eine Matrix $(h_{\sigma s})$ beschrieben, wobei sich der griechische Index auf das Lokal-System, der lateinische auf das Gauss'sche Koordinatensystem bezieht.[1]

Während nun der Vektor (a) gemäss der Relation

$$a^\sigma = h^\sigma{}_s\, a^s, \qquad (a^s = h_\sigma{}^s\, a^\sigma)$$

nach Belieben sowohl auf das Gauss'sche System als auch auf das Lokal-System bezogen werden konnte, schien es bisher, dass der Semi-Vektor *nur* inbezug auf das lokale System beschrieben werden könne, nicht aber *unmittelbar* inbezug auf das Gauss'sche System.

In der vorliegenden Arbeit wird nun gezeigt, dass Semivektoren auch direkt auf das Gauss'sche System bezogen werden können. Dies geschieht durch Grössen, welche sich bezüglich der Koordinaten-Transformationen wie gewöhnliche Vektoren verhalten, aber durch ein anderes Gesetz der Differentiation gekennzeichnet sind. Es zeigt sich dabei, dass dies Differentiationsgesetz nicht durch die Metrik allein, sondern auch noch durch einen arbiträr angenommenen "Fern-Parallelismus" bestimmt wird, von dessen Wahl der Inhalt irgend welcher Differential-Relationen, z. B. der Dirac-Gleichungen aber de facto unabhängig ist.

Wir wollen keineswegs behaupten, dass die hier entwickelte Darstellung derjenigen vorzuziehen sei, welche durch explizite Einführung des orthogonalen Vierbeins $(h_{\sigma s})$ charakterisiert ist. Es ist aber interessant, an einem Beispiel darzutun, dass eine Grösse nicht nur durch ihr Transformationsgesetz in ihrem Wesen bestimmt ist, sondern ebensowohl z. B. durch ihr Gesetz der absoluten Differentiation.

In unserer Arbeit "Semi-Vektoren und Spinoren" haben wir (im §8) beim Einbau dieser Grössen in den R_4 der allgemeinen Relativitätstheorie ein normi-

[1] Semi-Vektoren und Spinoren, Berlin, Akademieberichte 1932.

ertes Vier-Bein $h_{\sigma i}$ eingeführt, das dem Lokalvektor ρ^σ den gewöhnlichen Raumvektor ρ^s vermöge

$$\rho_s = h_{\sigma s}\,\rho^\sigma, \qquad \rho_\sigma = h_{\sigma s}\,\rho^s \tag{1}$$

zuordnet. Wie in der zitierten Arbeit, deren Kenntnis vorausgesetst wird, beziehen sich griechische Indizes auf das normierte lokale Bein, die lateinischen auf die Koordinaten des R_4.

Es gelten für das Bein $h_{\sigma s}$ die normierenden Relationen

$$h^\sigma{}_s\,h_{\sigma t} = g_{st}, \qquad h_{\sigma s}\,h_\tau{}^s = g_{\sigma\tau} \tag{2}$$

$$g_{\sigma\tau} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{2'}$$

Ebenso wie dem Lokal-Vektor können wir dem (Lokal)-Semi-Vektor (erster Art) $\psi^{\bar\sigma}$ mit einem (komplexen) normierten 4-Bein $k_{\bar\sigma\bar s}$ einen Raum-Semi-Vektor $\psi_{\bar s}$ vermöge

$$\psi_{\bar s} = k_{\bar\sigma\bar s}\,\psi^{\bar\sigma}, \qquad \psi_{\bar\sigma} = k_{\bar\sigma\bar s}\,\psi^{\bar s} \tag{3}$$

zuordnen.

In Bezug auf die Koordinatentransformation verhalte sich $k_{\bar\sigma\bar s}$ und somit $\psi_{\bar s}$ wie ein gewöhnlicher Vektor, bei Beinwechsel

$$h'_{\sigma s} = a_\sigma{}^\tau\,h_{\tau s}, \qquad a_\sigma{}^\tau, \qquad \text{Lorenzdrehung,} \tag{4}$$

transformiere sich $k_{\bar\tau\bar s}$ nach

$$k'_{\bar\sigma\bar s} = b_\sigma{}^\tau\,k_{\bar\tau\bar s}, \tag{5}$$

wo

$$a_\sigma{}^\tau = b_\sigma{}^\rho\,\bar b_\rho{}^\tau \tag{6}$$

die Zerlegung der Lorenzdrehung in zwei solche von b, $\bar b$ Charakter ist. Daraus folgt, dass der Raum-Semi-Vektor $\psi_{\bar s}$ bei dieser Beinänderung ungeändert bleibt.

Wegen der vorausgesetzten Normiertheit des Beines $k_{\bar\sigma\bar s}$ gilt für den Raum-Semi-Masstensor

$$g_{\bar s\bar r} = k_{\bar\sigma\bar s}\,k_{\bar\rho\bar r}\,g^{\bar\sigma\bar\rho} \tag{7}$$

$$g_{\bar\sigma\bar\rho} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \tag{7'}$$

d.h. es ist

$$g_{\bar s\bar r} \equiv g_{sr}; \tag{8}$$

der Masstensor g_{sr} misst auch die Semi-Grössen.

In bezug auf das Verhalten bei Koordinatenwechsel besteht zwischen Raum-Semi-Grössen und gewöhnlichen Vektoren bei dieser Art der Beschreibung gar kein Unterschied. Ein solcher tritt erst auf bei der Bildung der invarianten Ableitung. Wie auch die Drei-Indizes Symbole dieser Ableitung gewählt sind, folgt aus (3)

$$\psi_{\bar{\sigma}\,;\tau} = k_{\bar{\sigma}\bar{s}\,;\tau}\,\psi^{\bar{s}} + k_{\bar{\sigma}\bar{s}}\,\psi^{\bar{s}}{}_{;\tau} \tag{9}$$

Nun wollen wir vom Semi-Vektor ψ schlechtweg reden können, ko-resp. kontravariant, Lokal-resp. Raum Indizes seien nur verschiedene Möglichkeiten seiner Darstellung. Dann aber müssen $\psi_{\bar{s};\tau} = 0$ und $\psi^{\bar{\sigma}}{}_{;\tau} = 0$ völlig äquivalente Aussagen sein. Das ist aber dann und nur dann der Fall, wenn

$$k_{\bar{\sigma}\bar{s}\,;\tau} = k_{\bar{\sigma}\bar{s}\,,\tau} - \Gamma^{\bar{r}}{}_{\bar{s}\tau}\,k_{\bar{r}\bar{s}} - \Gamma^{\bar{w}}{}_{\bar{s}\tau}\,k_{\bar{\sigma}\bar{w}} = 0. \tag{10}$$

Da (zitierte Arbeit S. (24)) die Drei Indizesgrössen $\Gamma^{\bar{r}}{}_{\bar{s}\tau}$ des Lokal Semi-Vektors bekannt sind, ergibt (10) die entsprechenden Grössen $\Gamma^{\bar{w}}{}_{\bar{s}\tau}$ des Raum Semi-Vektors. Aus (10) ersieht man, dass diese Grössen von einer Bein-Änderung (5) nicht berührt werden.[2]
Aus (10) und (7) folgt jetzt

$$g_{\bar{s}\bar{r}\,;k} = g_{\bar{s}r\,,k} - \Gamma^{\bar{p}}{}_{\bar{s}k}\,g_{\bar{p}\bar{r}} - \Gamma^{\bar{p}}{}_{\bar{r}k}\,g_{\bar{s}\bar{p}} = 0. \tag{11}$$

Da wegen (8) auch

$$g_{sr;k} = g_{sr,k} - \left\{{p \atop s\,k}\right\} g_{pr} - \left\{{p \atop r\,k}\right\} g_{sp} = 0 \tag{11'}$$

gilt, so ist (Differenz der zwei obigen Gleichungen)

$$\left\{{p \atop s\,k}\right\} - \Gamma^{p}{}_{sk} = A^{p}{}_{sk} \tag{12}$$

in den Indizes p, s schiefsymmetrisch.

Wir haben in der zitierten Arbeit besondere schiefsymmetrische Semi-Tensoren eingeführt $v_{\bar{\sigma}\bar{\tau}}$, die numerisch invariant blieben gegen Beindrehung, und deren definierende Relation

$$v_{\bar{\sigma}\bar{\tau}} = -\frac{i}{2}\,\eta_{\bar{\sigma}\bar{\tau}\bar{\kappa}\bar{\lambda}}\,v^{\bar{\kappa}\bar{\lambda}} \tag{13}$$

ist. Die Semi-Vektoren Ableitung war dabei so gewählt, dass

$$v_{\bar{\sigma}\bar{\tau}\,;\tau} = 0 \tag{14}$$

gilt für von Punkt zu Punkt konstante $v_{\bar{\sigma}\bar{\tau}}$.

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[2] Ist $\psi_{\bar{\sigma}}{}' = b_{\sigma}{}^{\tau}\psi_{\bar{\tau}}$ die Transformation des Semi-Vektors bei Beindrehung, so gibt $b_{\sigma}{}^{\tau}{}_{;\tau} = 0$ das Transformations gesetz für die Drei Indizes Symbole der gedrehten Beine. Bezeichnet man eine Ableitung $k_{\bar{\sigma}\bar{s}/\tau} = k_{\bar{\sigma}\bar{s},\tau} - \Gamma^{\bar{r}}{}_{\bar{s}\tau}\,k_{\bar{r}\bar{s}}$, so folgt aus (α) $\cdot\cdot$ $k'_{\bar{\sigma}\bar{s}} = b_{\sigma}{}^{\tau}\,k_{\bar{r}\bar{s}}$ stets $k'_{\bar{\sigma}\bar{s}/\tau} = b^{\tau}_{\sigma}k_{\bar{r}\bar{s}/\tau} + b^{\tau}_{\sigma/\tau}k_{\bar{r}\bar{s}}$. Nun ist $b^{\tau}_{\sigma/\tau} \equiv b_{\sigma}{}^{\tau}{}_{;\tau} = 0$. Also gilt mit (α) zugleich $k'_{\bar{\sigma}\bar{s}/\tau} = b^{\tau}_{\sigma}k_{\bar{r}\bar{s}/\tau}$, woraus aber die obige Behauptung folgt.

Wir fragen nach den (13), (14) entsprechenden Relationen für die zugeordneten Raum Semi-Tensoren

$$v_{\bar{h}\bar{j}} = k^{\bar{\sigma}}{}_{\bar{h}}\, k^{\bar{\tau}}{}_{\bar{j}}\, v_{\bar{\sigma}\bar{\tau}}, \qquad (v^{\bar{\kappa}\bar{\lambda}} = k^{\bar{\kappa}}{}_{\bar{\tau}}\, k^{\bar{\lambda}}{}_{\bar{s}}\, v^{\bar{\tau}\bar{s}}) \tag{15}$$

Wir multiplizieren (13) mit $k^{\bar{\sigma}}{}_{\bar{h}}\, k^{\bar{\tau}}{}_{\bar{j}}$ und erhalten[3]

$$\begin{aligned} v_{\bar{h}\bar{j}} &= -\frac{i}{2}\, \eta_{\bar{\sigma}\bar{\tau}\bar{\kappa}\bar{\lambda}}\, k^{\bar{\sigma}}{}_{\bar{h}}\, k^{\bar{\tau}}{}_{\bar{j}}\, k^{\bar{\kappa}}{}_{\bar{\tau}}\, k^{\bar{\lambda}}{}_{\bar{s}}\, v^{\bar{\tau}\bar{s}} \\ &= -\frac{i}{2}\sqrt{-g}\, \eta_{\bar{h}\bar{j}\bar{\tau}\bar{s}}\, v^{\bar{\tau}\bar{s}} = -\tfrac{1}{2}\sqrt{g}\, \eta_{\bar{h}\bar{j}\bar{\tau}\bar{s}}\, v^{\bar{\tau}\bar{s}} \end{aligned} \tag{16}$$

Die Relation (16) hat denselben Charakter als die Relation (13), ebenso folgt aus (15) wegen (10)

$$v_{\bar{h}\bar{j};\,r} = k^{\bar{\sigma}}{}_{\bar{h}}\, k^{\bar{\tau}}{}_{\bar{j}}\, v_{\bar{\sigma}\bar{\tau};\,r}, \tag{17}$$

also $v_{\bar{h}\bar{j};\,r} = 0$ für jene $v_{\bar{\sigma}\bar{\tau}}$ für die (14) gilt.

Bis jetzt haben wir die Wahl des $k_{\bar{\sigma}\bar{s}}$ Beins, das dem Lokal Semi-Vektor einen Raum Semi-Vektor zuordnet offen gelassen, von dieser Wahl abhängig ist natürlich der zugeordnete Raum Semi-Vektor, der bei Bein-Änderung eine bestimmte Drehung erfährt, deren spezielle Natur wir später diskutieren werden.

Wenn wir nun für den $E_{\alpha\bar{\sigma}\bar{\tau}}$ zugeordneten Raumtensor $E_{i\bar{s}\bar{l}}$ einen einfachen Ausdruck finden wollen, so ist es zweckmässig das $k_{\bar{\sigma}\bar{s}}$ Bein in besonderer Weise zu wählen und zwar soll bei Vornahme einer besonderen Beindrehung (4), (5) das h Bein mit dem k Bein zur Deckung gebracht werden können.

Wir können dann diesen Fall

$$(\alpha)\ \cdots \qquad h_{\sigma s} = k_{\bar{\sigma}\bar{s}}$$

für die Rechnung benutzen, da ja die Raumgrössen (Vektor, Semi-Vektor) nach ihrer Bildung bei weiterer Drehung der Beine (die die Relation (α) natürlich zerstört) unverändert bleiben. Ehe wir das $E_{i\bar{s}\bar{l}}$ berechnen, wollen wir dem in $E_{\alpha\bar{\sigma}\bar{\tau}}$ auftretenden Pseudo Vektor a^{τ} einen Raum Pseudo Vektor a_t vermittels

$$a_{\bar{l}} = h_{\bar{\tau}\bar{l}}\, a^{\bar{\tau}}, \qquad h_{\bar{\tau}\bar{l}} = h_{\tau t} \tag{18}$$

zuordnen. Das $\sim$ Zeichen über den Indixes erinnere daran, dass der Pseudo Vektor $a^{\bar{\tau}}$ bei der Beindrehung (6) keine Änderung erfährt: $a^{\bar{\tau}'} = a^{\bar{\tau}}$, d.h. sich verhält wir ein (geordnetes) System von vier Invarianten. Wir schreiben den Tensor $E^{\bar{\tau}\bar{\sigma}\alpha}$ in der Form

$$E^{\alpha\sigma\tau} = E^{\alpha\sigma\tau\rho}\, a_{\rho}, \tag{19}$$

wo

$$E^{\alpha\sigma\tau\rho} = g^{\alpha\sigma}\, g^{\tau\rho} + g^{\alpha\tau}\, g^{\sigma\rho} - g^{\alpha\rho}\, g^{\sigma\tau} + i\, \eta^{\alpha\sigma\tau\rho}. \tag{20}$$

[3] Es ist $|k^{\sigma}{}_{h}|$ stets $\pm\sqrt{-g}$, die Wurzel positiv. Wir treffen die Beinwahl so, dass diese Determinante $+\sqrt{-g}$ ist.

Nun ist $E_{\bar{\imath}\bar{\kappa}\bar{\lambda}} = h_{\alpha\bar{\imath}}\, k_{\bar{\sigma}\bar{\kappa}}\, k_{\bar{\tau}\bar{\lambda}}\, E^{\alpha\bar{\sigma}\bar{\tau}}$, also wegen (α)

$$(21)\qquad \begin{cases} E_{\bar{\imath}\kappa\lambda} = h_{\alpha\bar{\imath}}\, h_{\sigma\kappa}\, h_{\tau\lambda}\, E^{\alpha\sigma\tau} = h_{\alpha\bar{\imath}}\, h_{\sigma\kappa}\, h_{\tau\lambda}\, h_{\rho\tau}\, E^{\alpha\sigma\tau\rho}\, a^{\tau} \\ \qquad = E_{\bar{\imath}\kappa\lambda\tau}\, a^{\tau}, \end{cases}$$

wo

$$(21')\qquad E_{\bar{\imath}\kappa\lambda\tau} = g_{\bar{\imath}\kappa}\, g_{\lambda\tau} + g_{\bar{\imath}\lambda}\, g_{\kappa\tau} - g_{\bar{\imath}\tau}\, g_{\kappa\lambda} - \sqrt{g}\, \eta_{\bar{\imath}\kappa\lambda\tau}$$

ist.

In (21), (21′) erscheint der Raum-E-Tensor in derselben Art abgeleitet aus der Raummetrik wie in (19), (20) der Lokal-E-Tensor aus der Lokalmetrik. Aus (21) folgt weiter $E_{\bar{\imath}\bar{\kappa}\bar{\lambda};\tau} = 0$. Da der Lokal Pseudo-Vektor bei Bein (u. Koordinaten) Änderung invariant bleibt, so können wir seine invariante Ableitung ohne Benützung von (von Null verschiedenen) Drei-Indizes Grössen definieren

$$(22)\qquad a^{\bar{\tau}}{}_{;\tau} = a^{\bar{\tau}}{}_{,\tau} = \frac{\partial a^{\bar{\tau}}}{\partial x_{\tau}}.$$

Aus (18) folgt nun für jede Wahl der Drei Indizes Symbole

$$(23)\qquad a_{\bar{\imath};\tau} = h_{\bar{\tau}\bar{\imath}}\, a^{\bar{\tau}}{}_{;\tau} + h_{\bar{\tau}\bar{\imath};\tau}\, a^{\bar{\tau}}.$$

Wir können den Pseudo-Vektor als Raum- oder Lokal-Pseudo Vektor beschreiben, wenn wir die Drei Indizesgrössen für den Raum-Pseudo-Vektor gemäss

$$(24)\qquad h_{\bar{\tau}\bar{\imath};\tau} = h_{\bar{\tau}\bar{\imath},\tau} - \Delta^{\bar{s}}{}_{\bar{\imath}\tau}\, h_{\bar{\tau}\bar{s}} = 0$$

wählen. Durch Multiplikation mit $h^{\tau k}$ erhalten wir diese Grössen aus (24) in der Gestalt

$$(25)\qquad \Delta^{\bar{k}}{}_{\bar{\imath}\tau} = h^{\bar{\tau}\bar{k}}\, h_{\bar{\tau}\bar{\imath},\tau}$$

der bekannten "Klammern" des durch ein h-Bein vermittelten Fernparallelismus. Statt (23) gilt

$$(25')\qquad a_{\bar{\imath};\tau} = h_{\bar{\tau}\bar{\imath}}\, a^{\bar{\tau}}{}_{;\tau}.$$

Ist $a^{\bar{\tau}}$ konstant, so ist $a_{\bar{\imath}}$ "fernparallel": $a_{\bar{\imath};\tau} = 0$.

Es bleibt jetzt als letzte Rechnung die Bestimmung der Drei Indizes Symbole $\Gamma^{\bar{p}}{}_{\bar{s}\tau}$ für die Ableitungen der Raum-Semi-Grössen. Wir verwenden dazu die Relationen

$$(26)\qquad h_{\tau\bar{\imath};k} = h_{\tau\bar{\imath},k} - \left\{ {r \atop \bar{\imath}\, k} \right\} h_{\tau r} - \mathrm{P}^{\sigma}{}_{\tau k}\, h_{\sigma\bar{\imath}} = 0$$

$$(26')\qquad h_{\bar{\tau}\bar{\imath};k} = h_{\bar{\tau}\bar{\imath},k} - \Gamma^{\bar{r}}{}_{\bar{\imath}k}\, h_{\bar{\tau}\bar{r}} - \Gamma^{\bar{\sigma}}{}_{\bar{\tau}k}\, h_{\bar{\sigma}\bar{\imath}} = 0$$

$$(26'')\qquad h_{\bar{\tau}\bar{\imath};k} = h_{\bar{\tau}\bar{\imath},k} - \bar{\Gamma}^{\bar{r}}{}_{\bar{\imath}k}\, h_{\bar{\tau}\bar{r}} - \bar{\Gamma}^{\bar{\sigma}}{}_{\bar{\tau}k}\, h_{\bar{\sigma}\bar{\imath}}.$$

Dabei ist $P^{\sigma}{}_{\tau k}$ das Drei Indixes Symbol für die Ableitung des Lokal-Vektors

und es gilt für diese in σ, τ schiefsymmetrische Grösse die u, v Zerlegung (zitierte Arbeit S. (24))

$$P_{\sigma\tau k} = \Gamma_{\sigma\tau k} + \bar{\Gamma}_{\sigma\tau k}. \tag{27}$$

Aus den beiden ersten Relationen (26) und aus (27) folgt

$$(\{ {}^{r}_{ik} \} - \Gamma^{\bar{r}}_{\bar{i}k})\, h_{\tau r} = -\,[P^{\sigma}{}_{\tau k} - \Gamma^{\bar{\sigma}}_{\bar{\tau}k}]\, h_{\sigma t} = -\,\bar{\Gamma}^{\sigma}{}_{\tau k}\, h_{\sigma t} \tag{28}$$

resp.

$$\{ {}^{s}_{ik} \} - \Gamma^{\bar{s}}_{\bar{i}k} = -\,\bar{\Gamma}^{\sigma}{}_{\tau k}\, h_{\sigma t}\, h^{\tau s} = V^{s}_{i(k)}. \tag{28'}$$

Da sich $\bar{\Gamma}^{\sigma}_{\tau k}$ in den Indizes σ, τ algebraisch wie ein V Tensor verhält, so gilt dies in bezug auf die entsprechenden Indizes s, t für die linke Seite von (28'), was durch die Bezeichnung $V^{s}_{i(r)}$ angedeutet ist.

Aus den zwei letzten Gleichungen (26) und aus (27) folgt weiter

$$2h_{\tau t,k} - (\Gamma^{\bar{r}}{}_{ik} + \bar{\Gamma}^{\bar{r}}{}_{ik})h_{\tau r} - P^{\sigma}{}_{\tau k}\, h_{\sigma t} = 0, \tag{29}$$

was wieder mit der ersten Gleichung (26) kombiniert

$$h_{\tau t,k} + [\{ {}^{r}_{ik} \} - \Gamma^{\bar{r}}_{\bar{i}k} - \bar{\Gamma}^{\bar{r}}_{\bar{i}k}]\, h_{\tau r} = 0 \tag{30}$$

gibt.

Aus (30) und (24) folgt endlich

$$[\{ {}^{r}_{ik} \} + \Delta^{\bar{r}}_{\bar{i}k} - \Gamma^{\bar{r}}_{\bar{i}k} - \bar{\Gamma}^{\bar{r}}_{\bar{i}k}]\, h_{\tau r} = 0, \tag{31}$$

also

$$\begin{cases} \{ {}^{s}_{ik} \} - \Delta^{\bar{s}}_{\bar{i}k} = [\{ {}^{s}_{ik} \} - \Gamma^{\bar{s}}_{\bar{i}k}] + [\{ {}^{s}_{ik} \} - \bar{\Gamma}^{\bar{s}}_{\bar{i}k}] \\ \qquad\qquad = V^{s}_{i(k)} + U^{s}_{i(k)}. \end{cases} \tag{32}$$

(32) stellt also die Zerlegung des in s, t schiefsymmetrischen Tensors $\{ {}^{s}_{ik} \} - \Delta^{s}_{ik}$ in den U und V-Bestandteil dar. Damit ist nach (28') in einfachster Weise die Grösse $\Gamma^{\bar{s}}_{\bar{i}k}$ mitbestimmt. Vom gewählten h-Bein tritt in die Rechnung also nur das Drei Indixes Symbol Δ^{s}_{ik} für den diesem Bein entsprechenden Fernparallelismus.

Auch er spielt keine Rolle in bezug auf die Struktur der Dirac-Gleichungen (Amsterdamer Akademie der Wissenschaften 1933). Das kommt daher, dass bei Änderung dieses Fernparallelismus die Semi und Pseudo Vektoren bestimmte Transformationen (Substitutionen) erleiden, wobei die Dirac Gleichungen erfüllt bleiben.

Wir wollen zum Schlusse die Substitutionen berechnen, die ein Raum, Pseudo resp. Raum Semi-Vektor erfährt, sobald das Lokalbein und damit der Fernparallelismus geändert wird. Wir verwenden dazu unsere Kenntnis vom Verhalten der entsprechenden Lokalgrössen. Bei einer Beindrehung

$$h'^{\sigma}{}_{e} = a^{\sigma}{}_{\tau}\, h^{\tau}{}_{s}, \tag{33}$$

wo a^{σ}_{τ} eine Lorenzdrehung ist und

$$a^{\sigma}_{\ \tau} = b^{\sigma}_{\ \lambda}\, \bar{b}^{\lambda}_{\ \tau} \tag{34}$$

die ihr entsprechende b, $\bar{b}$ Zerlegung, erfährt der Raumvektor

$$\rho_e = h^{\sigma}_{\ e}\, \rho_{\sigma} \tag{35}$$

keine Änderung. Der Raum Pseudo Vektor

$$a_{\tilde{\imath}} = h^{\sigma}_{\ e}\, a_{\tilde{\sigma}}, \qquad (a_{\tilde{\sigma}} = h_{\sigma}^{\ e}\, a_{\tilde{\imath}}) \tag{36}$$

erfährt wegen $a'_{\tilde{\sigma}} = a_{\tilde{\sigma}}$ die durch

$$a'_{\tilde{\imath}} = h'^{\sigma}_{\ e}\, a_{\tilde{\sigma}} = h'^{\sigma}_{\ e}\, h_{\sigma}^{\ k} a_{\tilde{k}} = A_e^{\ k}\, a_{\tilde{k}} \tag{37}$$

$$A_e^{\ k} = h'^{\sigma}_{\ e}\, h_{\sigma}^{\ k} = a^{\sigma}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\sigma}^{\ k} \tag{37'}$$

gegebene Lorenzdrehung. Ist jetzt

$$\rho_{\tilde{\imath}} = h^{\sigma}_{\ e}\, \rho_{\tilde{\sigma}} \tag{38}$$

ein Raum Semi-Vektor, so gilt für $\rho'_{\tilde{\imath}}$

$$\begin{aligned} \rho'_{\tilde{\imath}} &= h'^{\sigma}_{\ e}\, \rho'_{\tilde{\sigma}} = (a^{\sigma}_{\ \tau}\, h^{\tau}_{e})\, (b_{\sigma}^{\ \mu}\, \rho_{\tilde{\mu}}) = b^{\sigma}_{\ \alpha}\, \bar{b}^{\alpha}_{\ \tau}\, b_{\sigma}^{\ \mu}\, h^{\tau}_{\ e}\, \rho_{\tilde{\mu}} \\ &= \bar{b}^{\mu}_{\ \tau}\, h^{\tau}_{\ e}\, \rho_{\tilde{\mu}} = \bar{b}^{\mu}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\mu}^{\ k} \rho_{\tilde{k}} = \bar{B}_e^{\ k}\, \rho_{\tilde{k}}, \end{aligned} \tag{39}$$

wo

$$B_e^{\ k} = b^{\sigma}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\sigma}^{\ k} \tag{39'}$$

die der Lorenztransformation A_e^k (37′) zugeordnete B-Transformation ist. In der Tat folgt das aus

$$\begin{cases} B_e^{\ k}\, \bar{B}_k^{\ r} = (b^{\mu}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\mu}^{\ k})\, (\bar{b}^{\alpha}_{\ \beta}\, h^{\beta}_{\ k}\, h_{\alpha}^{\ r}) = b^{\beta}_{\ \tau}\, \bar{b}^{\alpha}_{\ \beta}\, h^{\tau}_{\ e}\, h_{\alpha}^{\ r} \\ \qquad = \bar{b}^{\alpha}_{\ \beta}\, b^{\beta}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\alpha}^{\ r} = a^{\alpha}_{\ \tau}\, h^{\tau}_{\ e}\, h_{\alpha}^{\ r} = A_e^{\ r}. \end{cases} \tag{40}$$

Die Relationen $A_e^k;\ r = 0$, $B_e^k;\ r = 0$ liefern zum Schluss die Transformation der Drei-Indizes Symbole der Ableitungen für den Pseudo-resp. Semi-Raum Vektor.

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INVERSION FORMULAE AND UNITARY TRANSFORMATIONS.

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We consider the Hilbert space $\mathfrak{H}_0$ consisting of all complex-valued functions $\subset L^2(0, \infty)$ with the inner product

$$(f, g) = \int_0^\infty f(x)\,\overline{g(x)}\,dx.$$

As usual, we call a linear transformation

$$g = Tf$$

unitary, if it is a one-to-one linear transformation of $\mathfrak{H}_0$ into itself which is isometric:

$$(Tf_1, Tf_2) = (f_1, f_2).$$

Plancherel's well-known theorem on trigonometrical inversion may be thus interpreted that each of the two pairs of inversion formulae

$$\int_0^x g(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1-\cos xy}{y} f(y)\,dy, \qquad \int_0^x f(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1-\cos xy}{y} g(y)\,dy$$

$$\int_0^x g(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin xy}{y} f(y)\,dy, \qquad \int_0^x f(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin xy}{y} g(y)\,dy$$

represents a unitary transformation (and its inverse).

Recently Watson[1] has discussed the formulae

$$\int_0^x g(x)\,dx = \int_0^\infty \frac{\chi(xy)}{y} f(y)\,dy, \qquad \int_0^x f(x)\,dx = \int_0^\infty \frac{\overline{\chi(xy)}}{y} g(y)\,dy,$$

in which $\chi(t)$ is an unspecified function; and he has found that the formulae represent a unitary transformation if, and only if, the function $\chi(t)$ satisfies the condition

$$\int_0^\infty \frac{\chi(ay)}{y}\,\frac{\overline{\chi(by)}}{y}\,dy = \min(a, b) \qquad a > 0, b > 0.$$

There are unitary transformations which cannot be represented by Watson's formulae; for instance, the identical transformation is of that kind. But we

1. G. N. Watson, General Transforms, Proc. London Math. Soc. (2) 35, (1933), 156; compare also E. C. Titchmarsh, A proof of a theroem of Watson, Journ. London Math. Soc. 8 (1933), 217, and M. Plancherel, Sur les formules de réciprocité du type de Fourier, ibid. 220.

are going to show that every unitary transformation may be represented by the more general formulae

$$(1)\qquad \int_0^x g(\xi)d\xi = \int_0^\infty \overline{k(x,y)}\, f(y)\, dy, \qquad \int_0^x f(\xi)d\xi = \int_0^\infty \overline{l(x,y)}\, g(y)\, dy,$$

with appropriate kernels $k(x, y)$, $l(x, y)$, which, for each x, are of integrable square in $0 < y < \infty$.

Our theorem runs as follows.

I. *Every unitary transformation is representable in the form* (1), *and if the formulae* (1) *represent a unitary transformation, then, for all* $a > 0, b > 0$

$$(2)\qquad \int_0^\infty k(a,y)\,\overline{k(b,y)}\, dy = \min(a,b),$$

$$(3)\qquad \int_0^\infty l(a,y)\,\overline{l(b,y)}\, dy = \min(a,b),$$

$$(4)\qquad \int_0^a l(b,y)\, dy = \int_0^b \overline{k(a,y)}\, dy.$$

II. *If two functions* $k(x, y)$, $l(x, y)$ *satisfy these conditions, then the formulae* (1) *represent a unitary transformation*

Proof of I. If we introduce the functions

$$f_a(y) = \begin{cases} 1 \text{ for } 0 < y \leqq a, \\ 0 \text{ for } a < y, \end{cases}$$

$$(5)\qquad g_a(y) = k(a,y),$$

$$(6)\qquad h_a(y) = l(a,y),$$

then the formulae (1) are equivalent to

$$(g, f_a) = (f, g_a) \qquad a > 0,$$

$$(f, f_a) = (g, h_a) \qquad a > 0.$$

Let

$$(7)\qquad g = Tf$$

be a unitary transformation. Then our relations are

$$(Tf, f_a) = (f, g_a),$$

$$(f, f_a) = (Tf, h_a),$$

and in order that these relations hold for all $f \subset \mathfrak{H}_0$ and some $a > 0$, it is necessary and sufficient that

$$(8)\qquad f_a = Tg_a,$$

$$(9)\qquad h_a = Tf_a.$$

Thus any unitary transformation (7) is representable in the form (1) with kernels

$$k(a, y) = T^{-1}f_a(y), \qquad l(a, y) = Tf_a(y), \tag{10}$$

and, conversely, if the formulae (1) represent the unitary transformation (7) then the kernels have the values (10).

From (8), (9) it easily follows that

$$(g_a, g_b) = (f_a, f_b),$$
$$(h_a, h_b) = (f_a, f_b),$$
$$(f_a, g_b) = (h_a, f_b),$$

and these relations are equivalent to (2), (3), (4).

Proof of II. We define the functions $g_a(y)$, $h_b(y)$ by (5), (6), and we consider the two transformations

$$g_a = U_0 f_a, \tag{11}$$

$$h_a = V_0 f_a. \tag{12}$$

Their domain ϑ consists of the elements f_a, and the conditions (2), (3) say that both transformations are isometric:

$$(U_0 f_a, U_0 f_b) = (f_a, f_b),$$
$$(V_0 f_a, V_0 f_b) = (f_a, f_b).$$

Any finite number of elements $\subset\vartheta$ are linearly independent, and their linear combinations are everywhere dense in $\mathfrak{H}_0$. Now any isometric transformation defined on such a domain ϑ can be extended into an isometric linear transformation defined throughout $\mathfrak{H}_0$. We denote the extensions by

$$g = Uf, \tag{13}$$

$$h = Vf. \tag{14}$$

The condition (4) can be written in the form

$$(V_0 f_b, f_a) = (f_b, U_0 f_a). \tag{15}$$

For each $b > 0$ this relation holds for all f_a. As the transformation (13) is the smallest closed linear extension of the transformation (11), it easily follows that we have the relation

$$(V_0 f_b, f) = (f_b, Uf) \tag{16}$$

for all $f \subset \mathfrak{H}_0$. From this we deduce, by the same argument, that the relation

$$(V\varphi, f) = (\varphi, Uf)$$

holds for any two elements f, φ. Putting $\varphi = U\psi$, where ψ is any element $\subset \mathfrak{H}_0$, we get

$$(VU\psi, f) = (U\psi, Uf) = (\psi, f),$$

$$(VU\psi - \psi, f) = 0,$$

$$VU\psi = \psi. \tag{17}$$

From (17) we conclude that U and V are inverse one-to-one transformations of $\mathfrak{H}_0$ into itself. But a one-to-one isometric linear transformation of the whole space into itself is unitary, so that U and V are inverse unitary transformations.

According to the first part of our theorem the unitary transformation

$$g = U^{-1}f$$

can be represented by formulae (1), and it is easily seen that the kernels $k(x, y)$, $l(x, y)$ are the given functions.

Remarks

Our propositions on unitary transformations in $\mathfrak{H}_0$ can be partly generalized if we make use of the notion of an adjoint transformation.

We denote by $f, g, \cdots$ elements of $\mathfrak{H}_0$, by

$$g = Tf \tag{18}$$

any linear transformation whose domain is $\mathfrak{H}_0$, and by T^* its adjoint transformation. We shall apply the following important theorem which was found independently by J. v. Neumann and J. D. Tamarkin.

LEMMA. *If T is defined throughout $\mathfrak{H}_0$, and if the domain of T^* is everywhere dense in $\mathfrak{H}_0$, then T and T^* are both bounded and defined everywhere.*

1. Let us assume that our transformation (18) is representable by a formula

$$\int_0^x g(\xi)\, d\xi = \int_0^\infty \overline{k(x, y)}\, f(y)\, dy. \tag{19}$$

Introducing the function (5), this is equivalent to

$$(Tf, f_a) = (f, g_a). \tag{20}$$

By the definition of T^*, we have

$$g_a = T^* f_a. \tag{21}$$

Hence T^* is defined for a set of elements which determines $\mathfrak{H}_0$. By the lemma, T and T^* are bounded and defined everywhere. Conversely, if T^* is defined everywhere in $\mathfrak{H}_0$, the relation (20) holds for the functions (21). Thus we have

THEOREM 1. *In order that a linear transformation T whose domain is $\mathfrak{H}_0$ be representable by a formula* (19), *it is necessary and sufficient that T and T^* are defined everywhere and bounded.*

2. It follows from (19), by differentiation, that

$$g(x) = \frac{d}{dx}\int_0^\infty \overline{k(x, y)}\, f(y)\, dy \tag{22}$$

(in the sense of Lebesgue). But if we integrate (22), we get only

$$\int_0^x g(\xi)d\xi + C_f = \int_0^\infty \overline{k(x, y)}\, f(y)\, dy, \tag{23}$$

where C_f is a constant depending on $f(y)$.

Now let us assume that a linear transformation (18) is representable in the form (22). We want to show that our conclusion concerning T and T^* still holds. In order to eliminate the constant C_f we deduce from (23) the relation

$$\int_{x_0}^{x_1} g(\xi)\, d\xi = \int_0^\infty \left(\overline{k(x, y)} - \overline{k(x_0, y)}\right) f(y)\, dy,$$

in which x_0 is some fixed positive number. For the functions

$$\varphi_a = f_a - f_{x_0},$$
$$\psi_a = g_a - g_{x_0},$$

we again have the relations

$$(Tf, \varphi_a) = (f, \psi_a),$$
$$\psi_a = T^*\varphi_a.$$

and, as the elements φ_a determine the whole space $\mathfrak{H}_0$, we can again apply the lemma.

THEOREM 2. *In theorem* 1 *the relation* (19) *can be replaced by the more general relation* (22).

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ZUFÄLLIGE BEWEGUNGEN

(Zur Theorie der Brownschen Bewegung)

VON A. KOLMOGOROFF

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In zwei früheren Arbeiten[1] (im folgenden als I und II zitiert) habe ich eine allgemeine Theorie der stetigen zufälligen Prozesse entwickelt. Es wurde dort unter sehr allgemeinen Voraussetzungen bewiesen, dass, wenn der Zustand eines physikalischen Systems in jedem gegebenen Zeitmoment mit Hilfe von n Parametern $x_1, x_2, \cdots, x_n$ vollständig bestimmt ist und diese Parameter sich mit der Zeit t stetig ändern,[2] so genügen die entsprechenden Verteilungsfunktionen der Fokker-Planckschen Differentialgleichung. Im allgemeinen Fall eines solchen zufälligen Prozesses sind die Zuwächse Δx_i der Parameter x_i von der Grössenordnung $(\Delta t)^{\frac{1}{2}}$. Folglich wächst im allgemeinen das Verhältnis $\Delta x_i : \Delta t$ mit $\Delta t \to 0$ über jede endliche Grenze, so dass man von keiner bestimmten Geschwindigkeit der Änderung von x_i sprechen kann. Wir wollen jetzt zeigen, wie man die allgemeine Theorie auf den Fall der *zufälligen Bewegungen* anwenden kann, wenn man voraussetzt, dass nicht nur die Koordinaten des Systems sondern auch ihre Ableitungen nach der Zeit sich stetig ändern.

Es seien $q_1, q_2, \cdots, q_n$ die Koordinaten eines Systems mit n Freiheitsgraden. Wir setzen voraus, es existiere, wenn q und $\dot{q}$ in einem Zeitmoment t bestimmt sind, eine bestimmte Wahrscheinlichkeitsdichte

$$G(t, q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t', q_1', \cdots, q_n', \dot{q}_1', \cdots, \dot{q}_n')$$

für die möglichen Werte q' und $\dot{q}'$ der Koordinaten des Systems und ihrer Zeitableitungen in jedem Zeitmoment $t' > t$. Es wird noch vorausgesetzt, dass G von dem Verhalten des Systems vor dem Momente t unabhängig ist.

Es ist natürlich vorauszusetzen, dass

$$E|\Delta q_i - \dot{q}_i \Delta t| = o(\Delta t), \tag{1}$$

$$E(\Delta q_i)^2 = o(\Delta t) \tag{2}$$

ist,[3] wobei $\Delta t = t' - t$ ist und E die mathematische Erwartung bezeichnet. Aus (1) und (2) folgt

$$E(\Delta q_i) = \dot{q}_i \Delta t + o(\Delta t), \tag{3}$$

[1] I. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann 104 (1931), pp. 415-458. II. Zur Theorie der stetigen zufälligen Prozesse, Math. Ann. 108 (1933), pp. 149–160.

[2] Wegen der genaueren Bedeutung der Stetigkeitsbedingung bei zufälligen Prozessen vgl. I, §13.

[3] Da Δq_i von der Grössenordnung Δt sein muss.

$$E(\Delta q_i \Delta q_j) \leq \sqrt{E(\Delta q_i)^2\, E(\Delta q_j)^2} = o(\Delta t). \tag{4}$$

In II ist unter einigen sehr allgemeinen Voraussetzungen bewiesen, dass ausser (2–4) noch die folgenden Relationen gelten:

$$E(\Delta \dot{q}_i) = f_i(t, q, \dot{q})\Delta t + o(\Delta t), \tag{5}$$

$$E(\Delta \dot{q}_i)^2 = k_{ii}(t, q, \dot{q})\Delta t + o(\Delta t), \tag{6}$$

$$E(\Delta \dot{q}_i \Delta \dot{q}_j) = k_{ij}(t, q, \dot{q})\Delta t + o(\Delta t), \tag{7}$$

wobei f und k stetige Funktionen sind. In manchen physikalischen Fragen sind die Voraussetzungen (5–7) auch direkt nachweisbar. Aus (2) und (6) folgt

$$E(\Delta q_i \Delta \dot{q}_j) \leq \sqrt{E(\Delta q_i)^2\, E(\Delta \dot{q}_j)^2} = o(\Delta t). \tag{8}$$

Bei einigen physikalisch ganz natürlichen Nebenvoraussetzungen folgt aus (2–8), dass G die Fundamentallösung der folgenden Differentialgleichung vom Fokker-Planckschen Typus ist:[4]

$$\frac{\partial g}{\partial t'} = -\Sigma\, \dot{q}_i' \frac{\partial}{\partial q_i'} g - \Sigma \frac{\partial}{\partial \dot{q}_i'}\{f_i(t', q', \dot{q}')g\} + \Sigma\Sigma \frac{\partial^2}{\partial \dot{q}_i' \partial \dot{q}_j'}\{k(t', q', \dot{q}')g\}. \tag{9}$$

Im Falle $n = 1$ hat man also die Gleichung

$$\frac{\partial g}{\partial t'} = -\dot{q}' \frac{\partial}{\partial q'} g - \frac{\partial}{\partial \dot{q}'}\{f(t', q', \dot{q}')g\} + \frac{\partial^2}{\partial \dot{q}'^2}\{k(t', q', \dot{q}')g\}. \tag{10}$$

Wenn f und k konstant sind, so findet man als die fundamentale Lösung von (10) den Ausdruck

$$g = \frac{2\sqrt{3}}{\pi k^2 (t' - t)^2} \exp\left\{ -\frac{[\dot{q}' - \dot{q} - f(t' - t)]^2}{4k\,(t' - t)} - \frac{3\left[q' - q - \frac{\dot{q}' + \dot{q}}{2}(t' - t)\right]^2}{k^3 (t' - t)^3} \right\}. \tag{11}$$

Man sieht also, dass hier $\Delta \dot{q}$ von der Grössenordnung $(\Delta t)^{\frac{1}{2}}$ wie bei allgemeinen zufälligen Prozessen ist. Für Δq erhält[5] man aber

$$\Delta q = \dot{q}\Delta t + O(\Delta t)^{\frac{1}{2}}. \tag{12}$$

Man könnte beweisen, dass die letzte Tatsache auch im Falle der allgemeinen Gleichung (9) gilt.

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[4] Wegen der Beweise vgl. II.

[5] Man hat das so zu verstehen, dass die Wahrscheinlichkeit der Ungleichung $|\Delta q - \dot{q}\Delta t| \geq K(\Delta t)$ bei einer genügend grossen Konstante K kleiner als ein beliebiges festes $\epsilon > 0$ ist und zwar gleichmässig in Δt.

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ON LOCALLY CONNECTED AND RELATED SETS

BY S. LEFSCHETZ

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In our investigations on the fixed point problem[1] we came across a noteworthy type of sets belonging to a class later investigated at length by K. Borsuk[2] under the name of *retracts*. We considered in various places in our book certain generalized locally connected sets and indicated in outline their bearing on the fixed point problem. Our chief object in the present paper is to study more completely the mutual relations of those two classes of sets, and incidentally to exhibit more clearly their relation to the fixed point problem.

Let A be a compact metric space. We call a complex K on A consisting partly of abstract, partly of singular cells a *semi-singular* complex K on A (a more precise definition will be found in No. 3). We shall show that all absolute neighborhood retracts A are intrinsically characterized by the following property (a sort of uniform local connectedness): for every ϵ there is a δ such that every finite semi-singular complex K on A of mesh $< \delta$ can be completed to form a singular complex of mesh $< \epsilon$ on A. The uniformity property is implicit in the fact that K is arbitrary on A. The simplest case of this important property is that in which we know only the vertices of a singular K, i.e., the only non-abstract cells are the zero-cells (these are always assumed already present), the abstract cells of positive dimension all being of diameter $< \delta$. Then it is possible to insert all the other cells so as to keep their diameters $< \epsilon$. Moreover this must be possible regardless of the dimension of K. The ϵ, δ relationship has then the uniformity property also with respect to this dimension. A closely related characterization may also be given for the absolute retracts. From this it follows readily that the homology groups of A are of the same type as those of a finite simplicial p-complex (p finite) and in particular that its Betti-numbers $R_q(A)$ are all finite, the numbers $R_q(A)$ corresponding to $q > p$ being zero. This enables us to complete a result of *Topology* regarding the fixed point problem.

Our treatment rests upon the following noteworthy result: given a closed subset A of a Hilbert parallelotope $\mathfrak{H}$, we can construct an infinite complex K and a deformation Π of $\mathfrak{H}$ into K such that $\Pi = 1$ on A and $\Pi(\mathfrak{H} - A) = K$.

1. We first recall the basic definitions and introduce a certain number of convenient abridged notations.

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[1] See our Colloquium Lectures on Topology, New York, 1930, p. 347. The general notations and terminology of this treatise will be adopted in the present paper.

[2] Notably in two papers in Fundamenta Matematicae, vol. 17 (1931), vols. 18 and 19 (1932) (= Borsuk I, II, III in the sequel).

Retracts. Let B be a self-compact metric set and let A be a closed subset of B. We say that A is a *retract* of B if there exists a continuous single-valued transformation (= c. s. v. t.) T of B into A which reduces to the identity on A ($T.x = x$ for $x \subset A$). The set A is an *absolute retract* (= AR) if it is a retract for every $B \supset A$. The set A is a *neighborhood-retract* (= NR) of B if A is a retract of some relatively open subset U ($\supset A$) of B. Finally, A is an *absolute neighborhood retract* (= ANR) if it is a neighborhood retract for every $B \supset A$. In particular, (Borsuk, III p. 223) *an* ANR (*an* AR) *is characterized by the property that its image on the Hilbert parallelotope* $\mathfrak{H}^3$ *is a neighborhood retract* (*a retract*) *for* $\mathfrak{H}$.

Local Connectedness. A self-compact metric set A is *locally p-connected* if, for every positive ϵ, there is a positive η such that any singular p-sphere H_p on A with diameter $< \eta$ bounds a singular cell E_{p+1} of diameter $< \epsilon$. To be precise, a singular sphere H_p is the image of a non-singular sphere H'_p under a certain c. s. v. t. T. The pair T, H'_p is not unique,[4] and the condition imposed is that among all the possible choices there is one in which H'_p is the point-set boundary of a certain non-singular cell E'_{p+1} and in which T is a transformation defined over $\bar{E}'_{p+1}$, where $TE'_{p+1} = E_{p+1}$, $TH'_p = H_p$, diam $E_{p+1} < \epsilon$.

A condition equivalent to the preceding one, but formulated with much less precision, is that H_p is ϵ-homotopic to a point on A.

We shall say that A is an LC^p-set if it is locally q-connected for every q such that $0 \leqq q \leqq p$. A set A which is locally connected for every p will be called an LC-set. The LC-sets are the same as the locally connected sets of *Topology*.

2. Let a simplex σ_q have for its singular image (continuous single-valued transform) the singular simplex σ'_q on A. The corresponding images of the various faces (vertices, one-cells, etc.) of σ'_q are the faces of σ'_q. With reference to the Note just quoted, a face σ'_r of σ'_q has a certain autonomous existence independently of σ_q in the following sense: if T is the transformation sending σ_q into σ'_q, σ'_r need not be considered as derived from any face of σ_q by means of T, but it may equally well be considered as derived from some other simplex σ^*_r by a transformation T^*, where $T^* = TU^{-1}$ and where U is a barycentric transformation of σ_r into σ^*_r. Thus σ'_r is a face of σ'_q provided that among all the transformations T^* corresponding to it there exists one like T, that is, a transformation of σ_q into σ'_q which carries a face σ_r of σ_q into σ'_r so that on σ_r we have $T = T^*U$, where U is a barycentric transformation of σ_r into σ^*_r. These remarks are necessary for a proper understanding of the sequel.

3. We recall, with Vietoris, that by an abstract ϵ p-simplex σ_p on A is meant a set of $p + 1$ points of A whose sum is of diameter $< \epsilon$. Similarly an abstract complex K on A is an abstract simplicial complex made up of abstract σ's on A.

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[3] This is the subset of Hilbert space defined by $0 \leqq x_n \leqq 1/n$, $n = 1, 2, \cdots + \infty$.

[4] See our recent note, Bull. Am. Math. Soc., vol. 39, (1933), pp. 124–129.

The *mesh* of K is then, as usual, the maximum diameter of its σ's (i.e., of the sets of their vertices).

Let K' be a subcomplex of K which includes all its vertices, and let K^* be a geometric realization of K (a polyhedral simplicial complex whose structure is that of K) with K'^* as the realization of K'. A singular image $\mathfrak{R}'$ of K'^* on A shall be called a *semi-singular* image of K. In relation to K the image $\mathfrak{R}'$ partakes of both the singular and the abstract complexes since some of the cells of K have singular images on A, others are represented merely by their vertices, and others by some singular faces and by their vertices. If we had a proper singular image $\mathfrak{R}$ of K and removed from it the images of the cells of $K - K'$, the residue would be $\mathfrak{R}'$.

Any cell σ of K is then represented on A in $\mathfrak{R}'$ by certain singular faces, some vertices, and possibly its own singular image. The sum of all these sets, or of as many as are present, is a certain point-set $\bar{\sigma}'$ of A, and the largest diam $\bar{\sigma}'$ is called the mesh of $\mathfrak{R}'$.

4. THEOREM I. *A is an LC^p when and only when for every positive ϵ there exists a positive η such that any semi-singular image K'_p of a K_p on A with mesh $< \eta$ can be completed to form a singular image of K_p on A with mesh $< \epsilon$.*

Important special case: the semi-singular image is an abstract complex with the same structure as K_p.

The condition of the theorem is sufficient. For in the statement K_p may *a fortori* be replaced by any subcomplex; consequently p may be replaced by any $q < p$. Consider, for any such q, a singular H_{q-1} on A with diameter $< \eta$. It may be regarded as the point-set image of an $F(\sigma_q)$ the images of whose faces make up a semi-singular image of $\bar{\sigma}_q$. By hypothesis the image may be completed by a singular σ_q of diameter $< \epsilon$ to form the singular image of $\bar{\sigma}_q$. Hence A is locally q-connected for every $q \leqq p$, that is, it is an LC^p.

We proceed to show that the condition of the theorem is necessary. This is trivial for $p = 0$ and hence we may assume $p > 0$. This being the case, let $\eta_q(\epsilon)$ be the number η entering into the definition of local q-connectedness. We shall choose successive constants $\zeta_0, \cdots, \zeta_q (q \leqq p)$, where $\zeta_i < \zeta_{i+1}$, $\zeta_q = \epsilon$, and the other ζ's are constrained to satisfy certain inequalities to be considered presently.

In the first place we require $\zeta_0 < \eta_0(\zeta_1)$ and in any case we impose the condition $\eta(\epsilon, p) \leqq \zeta_0$. Let x_0x_1 be an abstract one-cell of K'_p, that is to say, let x_0 and x_1 be the images of the end-points of a one-cell of K_p. By hypothesis $d(x_0, x_1) \leqq \zeta_0 < \eta_0(\zeta_1)$; hence we may join the two points on A by a singular one-cell σ'_1 with diameter $< \zeta_1$. That is, we may replace the abstract one-cells of K'_p by singular one-cells of diameters $< \zeta_1$. Consequently in the new complex, which we continue to call K'_p, all the one-cells will be singular and none will be abstract. Now let $x_0x_1x_2$ be an abstract two-cell of K'_p. Its one-faces $\sigma'^i_1 (i = 0, 1, 2)$ are all singular and their sum is of diameter $< 2\zeta_1$. A necessary condition that they bound a singular 2-cell of diameter $< \zeta_2$ is that

(1) $$\text{diam } (\sigma_1'^0 + \sigma_1'^1 + \sigma_1'^2) < \eta_1(\zeta_2).$$

This will certainly hold if

(2) $$2\zeta_1 < \eta_1(\zeta_2).$$

If (1) holds for all similar triples of singular one-cells, and this will certainly be true if (2) holds, we shall be able to insert all the requisite two-cells and have them be of diameters $< \zeta_2$. As to those already present, they have diameters $\leqq \eta(\epsilon, p) < \zeta_0 < \zeta_2$. Therefore all the singular two-cells will be of diameters $< \zeta_2$. The same reasoning may evidently be repeated: if the $m+1$ vertices of the simplexes $\sigma_{m-1}'^i$ are the vertices of a potential σ_m', we must have

(3) $$\text{diam } \Sigma\, \sigma_{m-1}'^i < \eta_{m-1}(\zeta_m),$$

and these inequalities will certainly hold relative to each σ_m' if we have

(4) $$2\zeta_{m-1} < \eta_{m-1}(\zeta_m),$$

although (4) may be stronger than necessary. Given ϵ and p we can find η and the ζ's so as to satisfy all the inequalities (4) for $m = 1, 2, \cdots, p$, and the ζ_0 thus obtained is a suitable $\eta(\epsilon, p)$. When the inequalities (4) hold, the same is true with regard to the weaker inequalities (3), and the successive faces may be inserted one by one: first all the one-faces, then all the two-faces, etc., and finally σ_q' itself. This proves the necessity of the condition of the theorem.

5. Now let $\eta(\epsilon)$ denote the greatest lower bound of the $\eta(\epsilon, p)$ of Theorem I for a given ϵ and all values of p. If $\eta(\epsilon) > 0$, i.e., if in Theorem I we can always choose a fixed $\eta(\epsilon, p)$ independent of p, we shall call A an LC-set. This implies that A is an LC^p-set for every p, and moreover that it is such a set in a certain uniform manner relative to p.

Suppose that A is an LC-set and let α be its diameter. The largest value that ϵ may take is α, but as $\epsilon \to \alpha$ it may well happen that η remains less than a certain $\beta < \alpha$. If, however, $\eta \to \alpha$ at the same time as ϵ, we shall call A an $\overline{\text{LC}}$-set. The basic theorem to be established in the sequel is

THEOREM II. *The classes LC and ANR, likewise the classes $\overline{LC}$ and AR are respectively identical ($LC = ANR$; $\overline{LC} = AR$).*

This proposition embodies an intrinsic characterization of absolute retracts and absolute neighborhood retracts. The proof will rest upon a lengthy discussion which we shall take up presently.

6. By a classical result of Urysohn's, a self-compact metric set A has a topological image A' on the Hilbert parallelotope $\mathfrak{H}$. It will be convenient to denote by $x, y, \cdots$ the points of $\mathfrak{H}$ whose coordinates are $(x_1, x_2, \cdots)$, $(y_1, y_2, \cdots)$, $\cdots$, and by $\mathfrak{H}_n$ the n-face of $\mathfrak{H}$ on which all coordinates $x_{n+i} = 0$. In particular, $\mathfrak{H}_0 = (0, 0, \cdots)$ is the origin.

By the *projection* of x onto the face $\mathfrak{H}_n$ we shall mean the point $(x_1, \cdots, x_n,$

$0, 0, \cdots$). The projection A^n of any set A onto $\mathfrak{H}_n$ is the set of the projections of all its points. We observe that A^n is closed if A is closed.

7. Now let A be the set of the theorem. The discussion to follow is of no interest when $A = \mathfrak{H}$, so that we may assume that A is a proper subset of $\mathfrak{H}$. As a consequence, from a certain ν on (kept fixed henceforth), A^ν will be a proper subset of $\mathfrak{H}_\nu$.

We subdivide $\mathfrak{H}_\nu$ simplicially into a complex $\mathfrak{H}'_\nu$ of mesh $\epsilon_\nu = \epsilon/2^\nu > 0$, then take the sum of the closed cells of the subdivision that meet A^ν, and erect on each of these cells a rectangular prism in the space of $\mathfrak{H}_{\nu+1}$, thus obtaining a certain closed convex complex $\mathfrak{H}'_{\nu+1}$ on $\mathfrak{H}_{\nu+1}$ which $\supset A^{\nu+1}$. The distance $d_{\nu+1} = d(A^{\nu+1}, \mathfrak{H}_{\nu+1} - \mathfrak{H}'_{\nu+1})$ is such that $0 < d_{\nu+1} < \epsilon_\nu$. We subdivide $\mathfrak{H}'_{\nu+1}$ simplicially into a complex of mesh $\epsilon_{\nu+1} < \frac{1}{2}d_{\nu+1} < \frac{1}{2}\epsilon_\nu$, which we still call $\mathfrak{H}'_{\nu+1}$, take the sum of the closed cells of $\mathfrak{H}'_{\nu+1}$ which meet $A^{\nu+1}$, erect rectangular prisms on them in the space of $\mathfrak{H}_{\nu+2}$, etc., and proceed indefinitely in this way with the following mild modification. Suppose that $\mathfrak{H}'_n$ has a cell of the form $\sigma\sigma'$, where σ' is a cell of $\mathfrak{H}'_{n+1}$ but σ is not. Through the subdivision of $\mathfrak{H}'_{n+1}$ into a complex of mesh $\epsilon_{n+1} < \frac{1}{2}d_{n+1}$ it may turn out that σ' is subdivided into simplexes σ'^i. We then replace $\sigma\sigma'$ by the sum of the simplexes $\sigma\sigma'^i$, thus preserving simpliciality throughout.

Ultimately we shall have a sequence of complexes $\mathfrak{H}'_\nu$, $\mathfrak{H}'_{\nu+1}$, $\cdots$ where $\mathfrak{H}'_n$ is a simplicial closed complex on $\mathfrak{H}_n$ which $\supset A^n$, which is coincident with $\mathfrak{H}_n$ for $n = \nu$, whose mesh $\epsilon_n < \epsilon/2^n$, and which is such that the projections of its points on $\mathfrak{H}'_{n-1}$ are not farther than $\frac{1}{2}\epsilon_{n-1}$ from A^{n-1}. Moreover, by construction, $\mathfrak{H}'_n \cdot \mathfrak{H}'_{n+1}$ coincides with a closed subcomplex of both $\mathfrak{H}'_n$ and $\mathfrak{H}'_{n+1}$. Therefore $J_n = \mathfrak{H}'_n - \mathfrak{H}'_n \cdot \mathfrak{H}'_{n+1}$ is an open subcomplex of $\mathfrak{H}'_n$ whose mesh $< \epsilon_n$ and the projections of whose points on $\mathfrak{H}_{n-1}$ are not farther than ϵ_{n-1} from A^{n-1}. Setting $J = J_\nu + J_{\nu+1} + \cdots$, we have $J_p \cdot J_q = 0$ for $p \neq q$, and J is a closed infinite simplicial complex—closed in the sense that if σ is any cell of J then every face of σ is likewise a cell of J.

8. Let ρ_n denote the square root of the remainder after n terms of the convergent series $\frac{1}{1^2} + \frac{1}{2^2} + \cdots$, and let y^n denote the projection of any point x on $\mathfrak{H}_n$. Taking $x \subset J_{n+1}$ we have $x = y^{n+1} \subset \mathfrak{H}'_{n+1}$ and $y^n \subset \mathfrak{H}'_n \cdot \mathfrak{H}'_{n+1}$. Let z^n be a point of A^n such that $d(y^n, A^n) = d(y^n, z^n)$. That a point such as z^n exists follows from the fact that A^n is closed. Let z be a point of A with projection z^n on A^n. We have $d(x, A) \leqq d(x, z)$ and hence

$$(5) \qquad d(x, A) = d(y^{n+1}, A) \leqq d(y^{n+1}, y^n) + d(y^n, z^n) + d(z^n, z)$$

$$\leqq \frac{1}{n+1} + \epsilon_n + \rho_n \leqq \frac{1}{n+1} + \frac{\epsilon}{2^n} + \rho_n,$$

a quantity which $\to 0$ with $1/n$. Therefore $J_n \to A$ with increasing n, while $J_n \cdot A = 0$, and hence $J \cdot A = 0$.

9. Consider a point x of $\mathfrak{H} - A$ and its projection y^n. Since all the points of $\mathfrak{H}'_{n+1}$ can be projected onto $\mathfrak{H}'_n$, whenever $y^{n+1} \subset \mathfrak{H}'_{n+1}$ then likewise $y^n \subset \mathfrak{H}'_n$. Hence there exists a largest n such that $y^{n+1} \subset \mathfrak{H}'_{n-1}$, for, if the contrary were true, (5) would hold for every n, and hence it would strictly be the case that $d(x, A) = 0$ and $x \subset A$. Since $y^{n+2} \not\subset \mathfrak{H}'_{n+2}$ we likewise have $y^{n+1} \not\subset \mathfrak{H}'_{n+2}$ and hence $y^{n+1} \subset J_{n+1}$. Thus for each x there exists a unique n, and hence a unique y^n, such that $y^n \subset J_n$. This y^n can be on no other J since no two J's intersect. Moreover, owing to (5), $d(x, y^n) \to 0$ as $x \to A$.

We now define P to be the transformation: $x \to y = y^n$. The point $y = Px$ is uniquely defined for every $x \not\subset A$ so that the transformation P is single-valued over $\mathfrak{H} - A$. It is evident that P reduces to the identity on J. Furthermore P is defined as the identity for any point $x \subset A$. Thus P has the following properties:

(a) It is single-valued on $\mathfrak{H}$.
(b) $P = 1$ on $A + J$; $P(\mathfrak{H} - A) = J$.
(c) $d(x, Px) \to 0$ as $x \to A$.

10. Unfortunately P is not continuous. To prove this, let U^n be the open region of $\mathfrak{H}$ whose projection onto $\mathfrak{H}_n$ has $\mathfrak{H}'_n$ for closure. Since $\epsilon_n < \frac{1}{2}d_n$, we have at once that $U^n \subset U^{n-1}$, so that the boundaries $F(U^n)$ do not intersect one another. If $x_0 \subset F(U^n)$, we have $y_0 = Px_0 = y_0^n$. However, in any vicinity of x_0 there are points x such that $y = Px = y^n$, and points such that $y = y^{n-1}$. For two points of these two distinct types the corresponding y's are at a distance apart which is as near as we please to $\delta = d(y_0^{n-1}, y_0^n)$, and δ may reach the value $1/n$ without, however, exceeding it. This is a measure of the discontinuity in y when x crosses $F(U^n)$ and we notice that $\delta \to 0$ with $1/n$.

We shall restore continuity in the transformation by approximating P in the vicinity of $F(U^n)$ by a new continuous transformation Π so chosen as to preserve the three properties (a), (b), (c). To do this we first subdivide $\mathfrak{H}'_{n+1}$ regularly and take the sum N of all the cells of the subdivision with a vertex on J_n. N is a polyhedral neighborhood of J_n on $\mathfrak{H}'_{n-1}$ with the following property: if $\sigma = \sigma'\sigma''$ is any simplex of N, where $\sigma' \subset J_n$ and σ'' has no vertex on J_n, then through every point of N there passes a unique segment $\xi'\xi''$ with ξ' on σ' and ξ'' on σ''.[5] Now if y^{n-1} coincides with the point ξ of σ we shall choose as the point $y = \Pi x$ the point $y'^n = (x_1, \cdots, x_{n-1}, x'_n, 0, \cdots)$ of $\mathfrak{H}'_n$, where $x'_n = \dfrac{\xi\xi''}{\xi'\xi''} \cdot x_n$ (the terms of the fraction are the lengths of the segments indicated). It follows that on σ the point y varies continuously with x and coincides with Px on σ' and σ''. This gives new points Πx on $\mathfrak{H}_n$ when $y_{n-1} \subset \sigma$, and their totality constitutes a convex cell whose dimension = dim $\sigma + 1$. The closure of this cell and of the similar cells for all σ's of N are added to J_n, the sum, J'_n, is subdivided simpli-

[5] This is the "normal neighborhood" property of *Topology*, p. 91.

cially into cells of diameters $< \epsilon_n$, and this is done in such manner that $J'_n \cdot J'_{n+1}$ coincides with subcomplexes of both J'_n and J'_{n+1}. Again setting

$$K_n = J'_n - J'_n \cdot J'_{n+1}, \qquad K = K_\nu + K_{\nu+1} + \cdots,$$

we have, as before, that K is a simplicial closed infinite complex with $K \cdot A = 0$.

11. Since $\epsilon_{n-1} < \frac{1}{2} d_{n-1}$, no cell of J_{n-1} has vertices on both J_n and J_{n-2}; hence $\bar{N} \cdot J \subset J_{n-1} + J_n$ and the modifications entailed in P in correlation with N concern only the points x whose $Px \subset J_{n-1}$. Since $J_p \cdot J_q = 0$ for $p \neq q$, the modifications are consistent throughout $\mathfrak{H} - A$. For any point x for which no modification is assigned we specify $\Pi x = Px$. Finally for $x \subset A$ we specify $\Pi x = x$. To sum up:

I. Π is continuous and single-valued on $\mathfrak{H}$;
II. $\Pi = 1$ on A;
III. $\Pi(\mathfrak{H} - A) = K$; $\Pi\, \mathfrak{H} = \bar{K} = K + A$.

The only question that may be raised concerns the continuity of Π; it is proved as follows:

(a) let x_0 be neither on A nor such that Px_0 is on an N; then, in a suitable vicinity of x_0, Π coincides with P. Since P is continuous, Π is also continuous;

(b) let $x_0 \subset N$; then in a small neighborhood of x_0, Πx is the point y'^n of No. 11, $\Pi x_0 = y'^n_0$, and $\Pi x \to \Pi x_0$ as $x \to x_0$; hence Π is continuous at x_0;

(c) let $x_0 \subset F(U^n)$, so that $\Pi x_0 = Px_0 = y^n_0 = y'^n_0$; a point x in a small vicinity of x_0 may be on U^n or on $\mathfrak{H}'_n - \bar{U}^n$. In the first case $\Pi x = y^n$, in the second, $\Pi x = y'^n$, and both have the same limit y^n_0 as $x \to x_0$;

(d) let $x_0 \subset A$; if $Px \subset J_n$ then $d(Px, \Pi x) = x'_n \leqq 1/n$; now as $x \to x_0$, $n \to \infty$, $d(Px, \Pi x) \to 0$, and, since $d(x, Px) \to 0$, it follows that $d(x, \Pi x) \to 0$ also. This means that as $x \to x_0$, $\Pi x \to \Pi x_0 = x_0$. Therefore Π is continuous at x_0 and consequently throughout $\mathfrak{H}$.

12. If we combine what we have already proved, we have the comprehensive and noteworthy

THEOREM III. *Given a closed subset A of the Hilbert parallelotope $\mathfrak{H}$, there exists a closed infinite simplicial complex K on $\mathfrak{H} - A$, and a continuous single-valued transformation Π over $\mathfrak{H}$ such that:* (a) *K has at most a finite number of cells exterior to any neighborhood U of A and their maximum dimension $\to 0$ as $U \to A$, while the diameters of the cells on $U \to 0$ at the same time;* (b) $\Pi(\mathfrak{H} - A) = K$; (c) $\Pi = 1$ *on A (all points of A are fixed points of Π).*

When $A = \mathfrak{H}$ the proposition is still true with $K = 0$; in this case $\Pi = 1$. If $\dim A = n$ is finite the set has a topological image A' on a parallelotope R of a bounded S_n.[6] The decomposition of mutually inclusive polyhedral neighborhoods of A' tending to A' into simplicial complexes of meshes $\to 0$ gives rise to a K_n analogous to K for which $\Pi = 1$ over the whole of R, so that the analogous theorem becomes trivial.

[6] See, for example, our paper in the Annals of Mathematics, vol. 31, (1930).

13. Returning to the situation of No. 11 and earlier, let us now introduce the hypothesis that A is an LC-set. We first define a single-valued transformation T of the vertices of K into points of A as follows: if x is any vertex of K then we choose for $x' = Tx$ any point of A such that $d(x, x') = d(x, A)$. Such a point always exists since A is closed. If x is a vertex of K_{n+1} we have (No. 8)

$$d(x, A) = d(y^{n+1}, A) \leqq d(y^{n+1}, y^n) + d(y^n, z^n) + d(z^n, z)$$

$$\leqq \frac{1}{n+1} + 2\epsilon_n + \rho_n.$$

Since mesh $K_{n+1} < \epsilon/2^{n+1}$, T transforms the $q+1$ vertices of any σ_q of K_{n+1} into $q+1$ points of A whose sum is of diameter $< 2/(n+1) + 5\epsilon/2^n + 2\rho_n = \lambda_n$, where $\lambda_n \to 0$ monotonely with $1/n$.

From the construction of K_n it follows that if any σ of K_n has vertices that do not belong to K_n, they must be vertices of K_{n+1}. The sum of these special σ's is an open subcomplex K_n'' of $\bar{K}_n$, and the σ's whose vertices are all vertices of K_n make up a closed subcomplex K_n' of K_n.

Now consider the function $\eta(\epsilon)$ of No. 5, take a sequence $\{\xi_n\}$ which approaches 0 monotonely, and set $\eta_m = \eta(\xi_m)$, so that η_m likewise $\to 0$. Let r be the smallest integer such that $\lambda_{r+1} < \eta_1$. For every m there exists an n such that $\lambda_n \leqq \eta_m$. Allowing for repetitions among the ξ's and η's, but not among the λ's, we may say that for every m we shall have $\lambda_{r+m} < \eta_m$, where still $\eta_m \to 0$ with $1/m$.

14. Now let $x^1, x^2, \cdots$ be the vertices of K'_{r+m}. It follows from Theorem I that there exists on A a singular image $\mathfrak{K}'_{r+m}$ of K'_{r+m} with Tx^i as the image of the vertex x_i and with mesh $\mathfrak{K}'_{r+m} < \xi_m$. We now define as transform by T of K'_{r+m} on A the c. s. v. t. whereby K'_{r+m} goes into $\mathfrak{K}'_{r+m}$.

Now suppose that $\sigma = \sigma'\sigma''$ is a cell of K''_{r+m}, with σ' a cell of K'_{r+m} and σ'' a cell of K'_{r+m+1}. Then σ' and σ'' have for images $T\sigma'$ and $T\sigma''$ which are singular cells of diameters $< \xi_m$. On the other hand, the vertices of σ have for their transforms a set of diameter $< \lambda_{r+m}$. Therefore diam $(T\sigma' + T\sigma'') < 2\xi_m + \lambda_{r+m} \to 0$ with $1/m$.

According to Theorem I, for m above a certain value we shall be able to join $\mathfrak{K}'_{r+m}$ and $\mathfrak{K}'_{r+m+1}$ by a singular image $\mathfrak{K}''_{r+m}$ of K''_{r+m} such that the transforms of the cells σ' and σ'' will be as before, and moreover that mesh $\mathfrak{K}''_{r+m} \to 0$ with $1/m$. We now extend T as previously to the cells of K''_{r+m}, so that for m sufficiently high it is defined over K_{r+m}, and we set $\mathfrak{K}_{r+m} = \mathfrak{K}'_{r+m} + \mathfrak{K}''_{r+m} = TK_{r+m}$.

15. We now have a transformation T defined over a certain set

$$K^{(p)} = K_p + K_{p+1} + \cdots$$

provided that p is sufficiently large, and it is a c. s. v. t. over $K^{(p)}$. Observe that when $x \subset K_n$, $n > p$,

$$d(x, Tx) < \lambda_n + \text{mesh } \mathfrak{K}_n + \text{mesh } K_n \to 0$$

with $1/n$. Therefore, if we extend again T to A by the condition that $T = 1$ on A, T is a continuous single-valued transformation of $K^{(p)} + A$ into A.

Consider now the transformation Π of No. 11. Since it is continuous and $= 1$ on A, any relative neighborhood of A with respect to $\Pi\mathfrak{H}$ will be the transform of a neighborhood of A on $\mathfrak{H}$. It follows that there exists an open set $U \supset A$ such that $\Pi U \subset K^{(p)}$. Consequently $T\Pi \cdot U \subset A$, $T\Pi = T = \Pi = 1$ on A, $T\Pi \cdot U = A$. *Therefore the* LC*-set* A *is an* NR *as to* $\mathfrak{H}$ *and hence it is an* ANR.

16. Conversely, suppose that A is an ANR, and hence an NR as to $\mathfrak{H}$, and let U be an open set $\supset A$ such that there exists a c.s.v.t. T of U into A which reduces to the identity on A. We shall show that A is an LC-set.

We first observe that there is an open set U' such that $U \supset \bar{U}' \supset A$, so that U' may replace U above. In other words, we may assume that in place of U we have $\bar{U}$ throughout. Now, *given any* ϵ, *there exists a* $\xi > 0$ *such that if* C *is any subset of* $\bar{U}$ *whose diameter* $< \xi$, *then diam* $TC < \epsilon$. Since A is closed it possesses a finite ϵ-covering by relatively open sets u^i. Since T is continuous and $= 1$ on A the sets $U^i = \bar{U}T^{-1}u^i$ are relatively open for $\bar{U}$ and $u^i = TU^i$. Moreover, since $T\bar{U} = A$, $\{U^i\}$ is a finite covering of $\bar{U}$ by relatively open sets. The characteristic constant[7] ξ of that covering has the required property. For if diam C $< \xi$, some $U^i \supset C$, $U^i \supset TC$, hence diam $TC <$ diam $u_i < \epsilon$.

17. Now that we have ϵ and ξ, let us consider a finite covering of A by spherical regions $\{\mathfrak{S}^i\}$ with diameters $< \xi$ and also $< d(A, \mathfrak{H} - U)$ whose centers are on A, and let η be the characteristic constant of the covering. Every subset C of U which meets A and whose diameter $< \eta$ will be on some $\mathfrak{S}^i$ and we shall have diam $TC < \epsilon$.

It is now a simple matter to show that ϵ and η are related as required by the condition for an LC^p-set no matter what may be the value of p, and hence that A is an LC-set. For, let K_p be an abstract simplicial complex and let $\mathfrak{K}'$ be a semi-singular image of K_p on A with mesh $< \eta$. The problem is to complete it to a singular image $\mathfrak{K}$ of K_p with mesh $< \epsilon$.

We first take $\bar{U}$ itself as our set A and construct a singular image $\mathfrak{K}^*$ of K_p on $\bar{U}$ by the process of No. 4 when modified so that each new singular cell introduced will lie on an $\mathfrak{S}^i$.

Given any set C on $\mathfrak{H}$, the intersection of all the *convex* sets of $\mathfrak{H}$ containing C is a convex set which we shall denote by (C). Clearly if $C' \subset C$ then $(C') \subset (C)$. We observe incidentally that diam $C =$ diam (C).

Now consider any potential cell σ_q of $\mathfrak{K}^*$, and suppose that among its faces it already possesses the realized singular cells $\sigma', \cdots, \sigma$ of $\mathfrak{K}'$ among which are

[7] Let A be a closed subset of a compact metric space. With any finite covering of A by open sets $\{U^i\}$, there are associated two positive constants α and β defined as follows: any set B which meets A and whose diameter $< \alpha$ is on some U^i; if $B \subset A$, diam $B < \beta$ and B meets a certain number of sets U^i, these sets intersect on A. We call the smaller of the two numbers α, β the *characteristic constant* of $\{U^i\}$.

included all the vertices of σ_q. If $\zeta = \Sigma\sigma^i$, (ζ) is on some $\mathfrak{S}^i$. Hence to show that σ_q can always be chosen so as to be on some $\mathfrak{S}^i$ it is sufficient to prove that it can be chosen on (ζ). Now the process of No. 4 consists in introducing first the singular one-cells, then all the singular two-cells, etc. It is therefore sufficient to show that if the condition $\sigma_q \subset (\zeta)$ has already been fulfilled for the dimensions $< q$, it can be fulfilled for q.

Consider again σ_q and let σ'^i be those of its singular faces already present of all dimensions up to and including $q-1$, where if ζ^i is the analogue of ζ for σ'^i we have $\sigma'^i \subset (\zeta^i)$. Since $\zeta^i \subset \zeta$, we have $(\zeta^i) \subset (\zeta)$, and hence $\Sigma\sigma'^i \subset (\zeta)$. Thus the singular boundary sphere η_{q-1} of σ_q is already introduced and it lies on (ζ). The segments from a fixed point x of (ζ) to the points of η_{q-1} generate a suitable $\sigma_q \subset (\zeta)$. This shows that we can construct σ_q as required.

The rest is now obvious. Take for $\mathfrak{R}$ the singular complex on A: $\mathfrak{R} = TK^*$. Since $\mathfrak{R}' \subset A$, $T\mathfrak{R}' = \mathfrak{R}'$, hence $\mathfrak{R}$ is a singular image of the type required. Moreover, since each cell of $\mathfrak{R}^*$ is on an $\mathfrak{S}^i$, mesh $\mathfrak{R}^* < \eta$, and hence mesh $\mathfrak{R} < \epsilon$. Therefore ϵ and η are related as required, and A is an LC-set.

18. We have thus completely identified LC-sets and ANR-sets. We shall now identify $\overline{\text{LC}}$-sets and AR-sets. Referring to No. 15 we find that in order that A be an AR it is sufficient that the transformation T be defined not merely on $K^{(p)}$ for p above a certain value, but on every $K^{(p)}$, that is, on K itself. This in turn will certainly occur provided that ϵ and $\eta(\epsilon)$ tend simultaneously to diam A, that is, provided that A is an $\overline{\text{LC}}$-set.

Conversely let A be an AR. Since an AR is also an ANR, A is certainly an LC-set. On the other hand the extension from the semi-singular $\mathfrak{R}'$ of No. 17 to the singular $\mathfrak{R}$ can be carried out here whatever $\mathfrak{R}$ may be. Therefore A is an $\overline{\text{LC}}$-set provided that there exists a $\mathfrak{R}'$ of mesh $\alpha =$ diam A. Since A is closed it contains two points x, y whose distance is α, and since A is a retract of $\mathfrak{H}$ there is a c. s. v. t. T of $\mathfrak{H}$ into A with $T = 1$ on A. The image under T of the closed segment xy is a closed singular $\sigma_1 \subset A$ whose diameter $= \alpha$, and σ_1 plus any point is precisely a suitable $\mathfrak{R}'$ of mesh α which is the semi-singular image of a closed two-cell. This shows that AR $= \overline{\text{LC}}$ and completes the proof of Theorem II.

19. **Homology characters of LC-sets.** Let A and U be as before and let us observe that there exists a constant $\alpha > 0$ such that if $d(x, A) < \alpha$, then the segment $\overline{x, Tx} \subset U$. For otherwise we could find a sequence of points x_i converging to a point x_0 such that $d(x_i, Tx_i) \to 0$ with $1/i$, and that for every i the segment $\overline{x_i, Tx_i}$ contains points of $\mathfrak{H} - U$. Therefore we would have $d(x_0, \mathfrak{H} - U) = 0$. But x_0 is the limit of the points Tx_i of A, and hence $x_0 \subset A$ since A is self-compact. We would then have $d(A, \mathfrak{H} - U) = 0$, which is ruled out by the fact that U is a neighborhood of the closed set A on the compact space $\mathfrak{H}$. We shall assume not only that α has the property just described, but that in addition it is $< d(A, \mathfrak{H} - U)$.

Let $\Sigma = \{U^i\}$ be a finite ϵ-covering of A by open sets of $\mathfrak{H}$, where $\epsilon < \alpha/8$ and where the covering is such that its skeleton Φ in the sense of Alexandroff is the same as for the aggregate $\{\bar{U}^i\}$.[8] We denote by ξ the characteristic constant of Σ, and assume Φ to be constructed as in *Topology*, p. 326, with the vertex corresponding to U^i not farther than $\alpha/8$ from the set. Consequently mesh $\Phi < \frac{1}{2}\alpha$; since each cell of Φ has a vertex not farther than $\alpha/8$ from A_1, every point x of Φ is nearer than α to A and the segment $\overline{x, Tx} \subset U$.

Since an LC-set is also an LC$^\infty$-set, or a locally-connected set in the sense of *Topology*, its homology groups defined by sequences of abstract chains are simply isomorphic with those defined by means of the singular chains on the set (*Topology*, p. 333, Remark I).

20. Let C_p be a singular chain on A with mesh $< \alpha/8$ and ξ, and let τ be a transformation of the vertices x^j of C_p into vertices of Φ such that τx^j is the vertex of Φ corresponding to one of the sets $U^i \supset x^j$. It follows from the construction of Φ that, to the vertices of a cell σ of C_p, there correspond the vertices of a simplex $\tau\sigma$ of Φ. From this we conclude, as in *Topology*, p. 86, that C_p may be β-deformed into a subchain C'_p of Φ, where $\beta < \alpha$, and that τ may be extended to a c. s. v. t. of C_p into C'_p, still to be called τ. Consequently if x is any point of C_p, the segment $\overline{x, \tau x} \subset U$ and the totality of these segments gives rise to a deformation $\mathfrak{D}$ of C_p into C'_p in the sense of *Topology*, p. 78, with a deformation-chain $\mathfrak{D}C_p \subset U$.

Since τ is a linear, boundary-preserving operation on chains, it transforms cycles into cycles. That is, if Γ^i_p are cycles of A, the chains $\tau\Gamma^i_p$ are cycles of Φ.

Suppose that we have

$$\Phi \supset C_{p+1} \rightarrow \Sigma\, s_i \cdot \tau\Gamma^i_p.$$

We then have successively:

$$U \supset \Sigma\, s_i \cdot \mathfrak{D}\Gamma^i_p \rightarrow \Sigma\, s_i\,(\tau\Gamma^i_p - \Gamma^i_p);$$

$$U \supset D_{p+1} = C_{p+1} - \Sigma\, s_i \cdot \mathfrak{D}\Gamma^i_p;$$

$$A \supset TD_{p+1} \rightarrow \Sigma\, s_i \cdot \Gamma^i_p \sim 0 \text{ on } A.$$

Consequently any formal homology between the $\tau\Gamma$'s on Φ is also satisfied by the Γ's on A. Therefore the absolute homology groups of A are subgroups of those of the finite complex Φ, and hence they have the same general structure: the Betti-numbers and torsion coefficients are all finite, and those above a certain p are zero.[9] In particular, for an $\overline{\text{LC}}$-set we may take for Σ the single set $\mathfrak{H}$, so

[8] To have a suitable Σ we may, for example, take a first covering by spherical regions of radii $\frac{1}{2}\epsilon$, then replace these by concentric regions whose radii are slightly smaller.

[9] Added in proof, Jan. 5, 1934: I learn from a preprint of a paper by Borsuk, due to appear in Fundamenta Matematicae, vol. XXI, that he has proved the slightly more general result, that the homology groups are those of a finite complex. This may readily be obtained by a slight extension of our treatment, the complex being a suitable Φ.

that Φ is a single point and the homology characters are those of a point, that is, the set is *cell-like* in the sense of A. W. Tucker. Combining our results we have

THEOREM IV. *For every* LC-*set* (= ANR) *there exists an integer* p *such that all the absolute homology characters for the dimensions* $\leqq p$ *are finite and the others are zero. An* $\overline{\text{LC}}$-*set* (= AR) *is cell-like.*

The least integer p for which the preceding results hold is clearly a topological invariant of the LC-set A. We shall call it the *characteristic integer* of the set.

21. **Applications.** Taking account of Alexander's generalized duality relations (*Topology*, p. 339) we have, as a consequence of Theorem IV,

THEOREM V. *If* A *is an* LC-*set in a sphere* H_n, *the Betti-numbers of* $H_n - A$ *are all finite and* $p \leqq n$. *If* A *is an* $\overline{\text{LC}}$-*set,* $H_n - A$ *has the Betti-numbers of a point.*

As a special case, $R_0(H_n - A)$ is finite, that is, $H_n - A$ consists of a finite number of regions, and if A is an $\overline{\text{LC}}$-set, $H_n - A$ is connected (Borsuk III, p. 230).

Finally, if we recall that our basic fixed point formula is strictly applicable to an ANR-set (*Topology*, p. 347) we have

THEOREM IV. *Let* T *be a c.s.v.t. of an* LC-*set* A *into itself or into part of itself, and let* p *be the characteristic integer of* A. *If* φ^q *is the matrix of the transformation induced by* T *on a base for the* q-*cycles of* A, $\varphi^q = 0$ *for* $q > p$, *and* φ^q, *is finite for* $q \leqq p$. *The signed number of fixed points* (*Topology*, *p. 359*) *is given by a finite sum:*

$$\Theta = \Sigma\,(-1)^q \operatorname{trace} \varphi^q.$$

In particular,

$$\Theta = \operatorname{trace} \varphi^0 \neq 0 \text{ for an } \overline{\text{LC}}\text{-set.}$$

There is at least one fixed point when $\Theta \neq 0$. *In particular, every c.s.v.t. of an* $\overline{\text{LC}}$-*set has at least one fixed point.*

APPLICATION. Every acyclic continuous curve (= Peano continuum) is an $\overline{\text{LC}}$-set (Borsuk II, p. 211). Hence *every c.s.v.t. of a continuous curve into itself or into part of itself has at least one fixed point.* (See Borsuk II, p. 205.)

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ON THE HOMOLOGY GROUPS OF ABSTRACT SPACES

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1. **Introduction.** L. Pontrjagin has been kind enough to call my attention to an error in the concluding pages of a paper by L. W. Cohen and me on "A Classification of the Homology Groups of Compact Spaces."[1] The essential missing step in the argument will be supplied below. It will be shown, §24, that although the groups we failed to differentiate are non-countable topological groups with limit elements, their classification reduces, by an easy dualization, to the known classification of certain discrete, countable groups of quite another type.[2] This result is, however, incidental. The present paper is primarily a preliminary study of an important special class of topological groups, here described as *indexed groups*. These groups arise in a natural manner as the homology groups of abstract spaces that are not required to be either closed point sets or regions. A criterion for the homeomorphism of two such groups is given in §22 but a complete classification of the groups is not attempted.

2. **Indexed Groups.** We shall be dealing exclusively with commutative groups and shall, therefore, express the resultant of a pair of elements x and y as a sum $x + y$, rather than as a product xy. A linear form with integer coefficients,

$$\Sigma a_i x_i,$$

will designate the resultant of a set of elements x_i in which the element x_i appears the number of times indicated by the value of its coefficient.

An *indexed group* Γ will consist of an ordinary group G together with a correspondence associating with each element x of G a real number $N(x)$ subject to the following three conditions:

(i) $N(0) = 0$,
(ii) $N(x) > 0, \quad (x \neq 0)$,
(iii) If $N(x) > N(y)$, then $N(x + y) = N(x)$.

The number $N(x)$ will be called the *index*, or *norm*, of the element x. An indexed group Γ can be formed out of the elements of any ordinary group G by

[1] Annals of Mathematics, vol. 33 (July, 1932), pp. 538–566. The argument in §21, p. 563 is invalid. The group T_π cannot, in general, be resolved into a direct sum of cyclic groups and π-adic groups.

[2] H. Ulm, "Abzählbar-unendliche Abelsche Gruppen," Math. Ann., vol. 107 (1933), pp. 774–803.

defining the norm of the zero element of the group as 0 and the norm of every other element as 1. The following example is not quite so trivial. Suppose we take the group X of all finite linear combinations

$$\xi = \sum_{i=1}^{\alpha} a_i x_i \qquad (\alpha \text{ finite but not bounded})$$

of a countable set of marks x_i $(i = 1, 2, 3, \cdots)$. Then from the group X we can form an indexed group Ξ by defining the norm $N(0)$ of the zero element of X as 0 and the norm $N(\xi)$ of each other element ξ as the subscript n of the last non vanishing coefficient a_n in ξ. We can also form a totally different indexed group Ξ' by defining the norm of the zero element as 0 and the norm of every other element ξ as the reciprocal $1/m$ of the first non vanishing coefficient a_m in ξ. The reciprocal of m must be used, rather than m itself, in order that Condition (iii) be satisfied.

The homology groups of compact spaces are all special examples of indexed groups.[3]

3. *Let Ξ be an arbitrary indexed group and let n be an arbitrary positive number. Then the set X_n of all elements of Ξ of norms less than or equal to n determines an indexed subgroup Ξ_n of Ξ.*

To prove the theorem we first observe that an element and its negative always have the same norm:

$$N(x) = N(-x). \tag{3.1}$$

Otherwise, the norm $N(x - x)$ of the sum of the two elements would be equal to the greater of the two norms $N(x)$ and $N(-x)$, by (iii), and would therefore be greater than 0, by (ii), contrary to (i). Next, we observe that if x and y are any two elements of Ξ the norm $N(x + y)$ of their sum cannot be greater than both their norms $N(x)$ and $N(y)$. For if it is greater than $N(x)$,

$$N(x + y) > N(x) = N(-x),$$

then it must be equal to $N(y)$,

$$\begin{aligned} N(y) &= N[(x + y) - x], \\ &= N(x + y), \qquad \text{by (iii).} \end{aligned}$$

Thus the negative of every element of X_n is in X_n and the sum of every pair of elements of X_n is in X_n. Therefore X_n determines a subgroup Ξ_n of Ξ.

It is evident that the theorem still holds if we replace the words "less than or equal to n" by "less than n".

We see, as an immediate corollary, that the norm of a multiple nx of an element x can never exceed the norm of x itself:

$$N(nx) \leq N(x), \qquad (n \text{ any integer}). \tag{3.2}$$

[3] L. Vietoris, Über den höheren Zusammenhang Kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. Math. Ann., vol. 97 (1927), pp. 454–472.

There is nothing, however, to prevent the norm of nx from being smaller than the norm of x.

4. **Metrization.** An indexed group Ξ can always be thought of as a metric group in which the distance $D(xy)$ between two elements x and y is defined as the norm of their difference:

$$D(xy) = N(x - y). \tag{4.1}$$

By (4.1) and (3.1) the distance function is symmetrical,

$$D(xy) = D(yx). \tag{4.2}$$

Moreover, we readily see that it satisfies the triangle axiom,

$$D(xy) + D(yz) \geq D(xz), \tag{4.3}$$

so that the *group space of Ξ is a metric space.* Instead of proving (4.3) directly, we shall prove the following stronger theorem.

Let x, y and z be any three elements of Ξ written in such an order that we have

$$D(xy) \geq D(xz) \geq D(yz).$$

Then the distances $D(xy)$ and $D(xz)$ are always equal,

$$D(xy) = D(xz). \tag{4.4}$$

For we may write

$$D(xy) = N(x - y) = N[(x - z) + (z - y)].$$

Therefore, by the theorem of §3, $D(xy)$ cannot exceed both $D(x\,z) = N(x - z)$ and $D(yz) = N(z - y)$. It must, therefore, be equal to $D(xz)$.

The triangle formula (4.3), in which the elements x, y, and z appear in any order whatever, is an immediate consequence of the above theorem.

Now that we have defined distance we can immediately assign a meaning to such terms as *limit, convergence, continuity,* etc. We shall not insist on formal definitions since the terms will always be used with their customary meaning.

5. *The group space of an indexed group Ξ is of dimensionality zero.*

For let x be any element of the group Ξ and let n be any positive number. Moreover, let y be any element within a distance n of x and let z be any element at a greater distance than n from x. Then, by (4.4), we have

$$D(yz) = D(xz) > n.$$

In other words, no matter how small the positive number n may be, there is a neighborhood of the element x consisting of all points y within a distance n of x such that no point of this neighborhood is within a distance n of any point z of the residual space. The space is therefore totally disconnected, that is, it is of dimensionality zero.

6. **Homeomorphic Groups.** Two indexed groups Ξ and H will be said to be *homeomorphic* if there is a one-one correspondence between the elements x_i of Ξ and the elements y_i of H which is both simply isomorphic and bicontinuous. If Ξ and H are homeomorphic the correspondence $x_i : y_i$ will be called a *homeomorphic* correspondence. A correspondence $x_i : y_i$ is homeomorphic if, and only if, the following three conditions are satisfied:

(a) The negative of each element x_i is always paired with the negative of the element y_i corresponding to x_i.

(b) The sum of two elements x_i and x_j is always paired with the sum of the elements y_i and y_j corresponding to x_i and x_j respectively.

(c) A sequence of elements x_i converges to the zero element of Ξ if, and only if, the corresponding sequence of elements y_i converges to the zero element of H.

Conditions (a) and (b) state that the correspondence $x_i : y_i$ is simply isomorphic. Condition (c) immediately implies that the correspondence is bicontinuous.[4] Condition (c) may also be expressed by saying that

$$(c') \qquad \lim_{i \to \infty} N(x_i) = 0 \text{ if and only if } \lim_{i \to \infty} N(y_i) = 0.$$

Every indexed group Ξ is homeomorphic with an indexed group Ξ' such that the norm of each element of Ξ' other than the zero element is of the form $1/m$, where m is a positive integer.

The desired group Ξ' may be formed out of the elements x of Ξ by merely changing the norm $N(x)$ of each element to the greatest number $N'(x)$ of the form $1/m$ such that $N'(x) \leqq N(x)$. We can immediately verify that the new norms $N'(x)$ satisfy Conditions (i), (ii) and (iii) of §2, so that the group Ξ' is actually an indexed group. Moreover, the topological structure of the group Ξ' is the same as that of the group Ξ because a sequence of elements x_i, regarded as elements of Ξ, converges to zero if, and only if, this same sequence of elements, regarded as elements of Ξ', also converges to zero.

For the purposes of this paper, two homeomorphic groups will be thought of as essentially equivalent. Consequently we shall hereafter confine our attention to groups Ξ such that the norms of all their elements other than the zero element are of the form $1/m$, where m is any positive integer. The norm of the zero element must, of course, always be zero.

7. **Difference Groups.** Let Ξ be an arbitrary indexed group and H an arbitrary sub-group of Ξ. Then we may arrange the elements x of Ξ in co-sets $[x]$ in such a way that two elements x_i and x_j belong to the same co-set, $[x_i] = [x_j]$, if, and only if, their difference $x_i - x_j$ is an element of H. The co-sets $[x]$ are the elements of the so-called difference group Ξ (*mod* H). We shall prove that:

If the subgroup H *is closed with respect to the group* Ξ (in the sense that it con-

[4] Condition (a) may be shown to be a consequence of Conditions (b) and (c). It may therefore be dispensed with.

tains all its limit elements) *then the difference group* Ξ (*mod* H) *can be regarded as a well defined indexed group.*

We shall index the group Ξ (*mod* H) by assigning norms to the co-sets $[x]$ in the following manner. The co-set [0] containing the zero element will, of course, be the zero element of Ξ (*mod* H) and will therefore be given the norm 0. In every other co-set $[x]$ there will be an element x at a definite distance $1/m$ from the element 0 such that no other element of the co-set $[x]$ is nearer to the element 0 than $1/m$. (Otherwise, the element 0 would be a limit element of the set $[x]$ without actually belonging to $[x]$. Therefore the co-set $[x]$ would not be closed and it would follow at once that the co-set [0] would not be closed, contrary to the assumption that the subgroup H was closed.) The *norm* of the co-set $[x]$ will be defined as the number $1/m$. We can verify at once that the norms assigned to the co-sets $[x]$ satisfy Conditions (i), (ii), and (iii) of §2, so that the group Ξ (*mod* H) is an indexed group. The restriction that the subgroup H be closed is essential in order that there be a definite positive distance between co-sets, that is to say, in order that Condition (ii) be satisfied.

8. **Complete Groups.** Let Ξ be any indexed group, and let $\sigma(x_i)$ be any infinite sequence of elements x_i of Ξ, $(i = 1, 2, 3, \cdots)$. The sequence $\sigma(x_i)$ will be called a *Cauchy sequence* if the norm of the difference between two consecutive terms x_i and x_j converges to zero as i increases without bound,

$$\lim_{i\to\infty} N(x_{i+1} - x_i) = 0, \tag{8.1}$$

or, to state the condition in another way, if we have

$$\lim_{i\to\infty} (x_{i+1} - x_i) = 0. \tag{8.2}$$

Owing to the special character of the metric of Ξ, Condition (8.2) immediately implies the following one:

$$\lim_{i\to\infty} (x_{i+j} - x_i) = 0, \qquad (j > 0). \tag{8.3}$$

Our definition of a Cauchy sequence is therefore equivalent to that usually given.

Every convergent sequence is, of course, a Cauchy sequence, though in general the converse is not true. We shall say that the group Ξ is *complete* provided every Cauchy sequence of elements of Ξ converges to an element of Ξ.

Every indexed group Ξ *may be thought of as immersed in a well determined complete indexed group* Ξ^* *such that every element of* Ξ^* *is the limit of a convergent sequence of elements of* Ξ.

This theorem is hardly more than a special case of a well known theorem about abstract spaces, so that we shall merely outline its proof. We arrange all Cauchy sequences $\sigma(x_i)$ formed out of elements x_i of Ξ in sets $[\sigma(x_i)]$ such that two sequences $\sigma(x_i)$ and $\sigma(y_i)$ belong to the same set if, and only if, their difference $\sigma(x_i - y_i)$ converges to zero. Then we notice that in every Cauchy

sequence $\sigma(x_i)$ the norm of x_i approaches a definite limit as x_i increases without bound, and that this limit is the same for all sequences in the same set $[\sigma(x_i)]$. We call this limit the *norm* of the set:

$$N[\sigma(x_i)] = \lim_{i\to\infty} N(x_i).$$

The *negative* of a set $[\sigma(x_i)]$ is a well defined set $[\sigma(-x_i)]$ and the *sum* of two sets $[\sigma(x_i)]$ and $[\sigma(y_i)]$ is a well defined set $[\sigma(x_i + y_i)]$. The sets $\sigma(x_i)$ are thus the elements of a group Ξ^* which turns out to be the group for which we are seeking. The group Ξ^* is an indexed group, since the norm $N[\sigma(x_i)]$ obviously satisfies Conditions (i), (ii), and (iii) of §2.

To immerse the group Ξ in the group Ξ^* we have only to identify each element x of Ξ with the set $[\sigma]$ containing the sequence $x, x, x, \cdots$. Of course, there will generally be elements of Ξ^* that correspond to no element of Ξ. However, every such element $[\sigma(x_i)]$ will be the limit of a sequence of elements $[\sigma_i]$ identified with elements of Ξ. For example, we may define the set $[\sigma_i]$ as the one containing the sequence $x_i, x_i, x_i, \cdots$. The set $[\sigma_i]$ will then be identified with the element x_i of Ξ.

Finally, we must prove that the group Ξ^* is complete. To do this, we have to show that every Cauchy sequence

$$[\sigma(x_i^{(0)})],\ [\sigma(x_i^{(1)})],\ [\sigma(x_i^{(2)})],\ \cdots \tag{8.4}$$

of elements of Ξ^* converges to a limit $[\sigma]$. Now, by what we have just proved, the nth element $[\sigma(x_i^{(n)})]$ in (8.4) may be approximated to within a distance $1/n$ by an element $[\sigma_n]$ identified with an element x_n of Ξ. Since (8.4) is a Cauchy sequence it follows immediately that

$$[\sigma_1],\ [\sigma_2],\ [\sigma_3],\ \cdots$$

is also one and, therefore, that

$$x_1, x_2, x_3, \cdots \tag{8.5}$$

is one. The desired limit $[\sigma]$ of (8.4) is the class containing the sequence (8.5).

Of the two indexed groups Ξ and Ξ' given as examples in §2, the first is a discrete group and therefore automatically complete. The second is not complete, but may be completed by adjoining as new elements all infinite linear combinations of the elements x_i.

9. *If Ξ is a complete indexed group and* H *a closed subgroup of Ξ then the groups* H *and* Ξ (*mod* H) *are both complete.*

Proof. Every Cauchy sequence of elements of H must have a limit in Ξ, since Ξ is complete, and this limit must be in H, since H is closed. Therefore the group H is complete.

Next, we notice that if the distance between two co-sets $[x_i]$ and $[x_j]$ of Ξ (*mod* H) is D then we can choose the elements x_i and x_j in each of the co-sets respec-

tively so that the distance between them shall also be D. Moreover, if we replace x_i by any other element $x_i + y$ of the same co-set $[x_i]$ we can replace x_j by the element $x_j + y$ of the other co-set $[x_j]$ and again obtain two elements $x_i + y$ and $x_j + y$ at a distance D from one another. In other words, we can choose the element x_i in the first co-set at random and then pick an element x_j in the second which shall be at a distance D from x_i. Now, suppose

$$[x_1], [x_2], [x_3], \cdots \tag{9.1}$$

is any Cauchy sequence of elements Ξ (*mod* H). We may then choose the elements x_i, one after another, so that the distance of x_{i+1} from x_i shall be exactly the same as the distance of the co-set $[x_{i+1}]$ from the co-set $[x_i]$. The sequence

$$x_1, x_2, x_3, \cdots$$

will therefore be a Cauchy sequence of elements of Ξ converging to a limit element x. We see at once that the co-set $[x]$ containing the element x is the limit of the sequence (9.1). The completeness of the group Ξ (*mod* H) is thus established.

The subgroup Ξ_n of the group Ξ consisting of all elements of norms less than or equal to n, §3, is evidently closed. Therefore the groups Ξ_n and Ξ (*mod* Ξ_n) are complete.

From this point on we shall confine our attention to complete groups and their closed (and therefore complete) subgroups.

10. **Linear Forms.** Consider a set of elements x_α belonging to a complete group Ξ. We shall say that a linear form

$$\xi = \Sigma a_\alpha x_\alpha \tag{10.1}$$

in the elements x_α is *convergent* if, and only if, the number of elements x_α with non vanishing coefficients and of norm $1/m$ is finite for all positive values of m. Thus, if the form (10.1) is convergent it is either a finite form or one made up of a countably infinite set of terms. The case where the form is finite need not detain us. If the form is infinite we can choose our notation so that the elements x_α appearing in the form are denoted by x_i, $(i = 1, 2, 3, \cdots)$ respectively. We then see at once that the sum

$$s_n = a_1x_1 + a_2x_2 + \cdots + a_nx_n \tag{10.2}$$

of the first n terms of the form ξ converges to a definite limit element x as n increases without bound, and that this limit element x is independent of the way in which the terms x_i are ordered. We shall regard the form ξ as a representation of the limit element x.

At this point it is important to observe that the sum s_n may sometimes converge to a limit x even in cases where the form ξ contains an infinite number of elements x_i of the same norm. For the norm of a term a_ix_i may be smaller than that of the element x_i appearing in the term. We might, therefore, have

defined a convergent linear form in a different way, by merely stipulating that the number of *terms* $a_i x_i$ of the same norm be finite. The definition we have adopted corresponds to regarding a form as a sum of repeated *elements* x_i rather than as a sum of *terms* $a_i x_i$. It appears to be the more useful definition.

Given a set of convergent linear forms,

$$y_\alpha = \Sigma a_{\alpha\beta} x_\beta, \tag{10.3}$$

such that for every positive norm $1/m$ *there are at most a finite number of forms* y_α *involving elements* x_β *of norms greater than* $1/m$, *then every linear combination*

$$z = \Sigma b_\alpha y_\alpha \tag{10.4}$$

of the forms y_α is convergent. Moreover, we have

$$z = \Sigma b_\alpha y_\alpha = \Sigma (b_\alpha a_{\alpha\beta}) x_\beta. \tag{10.5}$$

The proof of this fundamental lemma is very obvious and will therefore be omitted. It should be noticed, however, that the phrase "elements x_β of norms greater than $1/m$" in the statement of the lemma cannot be replaced by the phrase "terms $a_{\alpha\beta} x_\beta$ of norms greater than $1/m$" which would have seemed the more natural one if we had adopted the alternative definition of a convergent linear form. For if the change of wording were made the forms y_α might together contain infinitely many terms $a_{\alpha\beta} x_\beta$ in the same element x_β, in which case the sum in the right hand member of (10.5) would no longer have any meaning.

As an immediate corollary to the lemma we see that:

Every finite linear combination

$$z = \sum_{i=1}^{n} b_i y_i$$

of a set of convergent linear forms

$$y_i = \sum_{\alpha} a_{i\alpha} x_\alpha$$

is a convergent linear form. Moreover, we have

$$z = \sum_{i=1}^{n} b_i y_i = \sum_{i\,\alpha} (b_i a_{i\alpha}) x_\alpha.$$

When we are dealing with systems of linear forms we shall often use the abridge matrix notation. For example, we shall express Relations (10.3) by a single matrix equation,

$$y = Ax,$$

and Relations (10.5) by

$$z = By = BAx.$$

This notation will presently be used with a still more general meaning.

11. **Bases.** A set of elements x_α belonging to an indexed group Ξ will be called a *basis* of Ξ provided it satisfies the following two conditions:

(a) Every element ξ of Ξ is representable as a convergent linear combination ax of the elements x_α (where we use the abridge notation ax for a linear form).

(b) If m is any positive integer the set of all elements of Ξ that are representable as convergent linear combinations of the elements x_α of norms less than $1/m$ is a neighborhood of the zero element of Ξ.

We wish to call particular attention to Condition (b). It implies that if m is an arbitrary positive integer there exists a positive integer $n(m)$ such that every element of Ξ of norm less than $1/n$ can be expressed as a convergent linear combination of base elements x_α of norms less than $1/m$.

A basis of Ξ will be called a *free basis* if there is only one linear combination of the base elements representing the element 0, namely, the one in which all the coefficients are zero. If the elements x_α form a free basis of Ξ then each element ξ of Ξ has one and only one representation as a linear combination ax of the base elements x_α. If it had more than one, the difference of any two of them would be a representation of 0 with non vanishing coefficients and the basis would not be a free basis.

A group Ξ will be called a *free group* if it has a free basis.

12. It will be convenient to designate the elements of a basis of Ξ by symbols with double subscripts, such as $x_{m\alpha}$. The first subscript m will be an integer and will indicate that the norm of $x_{m\alpha}$ is $1/m$. The second subscript α will be any distinguishing mark to differentiate elements of the same norm. We shall not require the second subscript to be an integer since we shall want to provide for the case where the base elements $x_{m\alpha}$ are not countable.

Every element ξ of the group Ξ will be representable by a convergent linear form ax,

$$\xi = ax. \tag{12.1}$$

Let us assume that the terms of ax are arranged in such an order that the norms of successive base elements $x_{m\alpha}$ in ax converge *steadily* to zero, by which we mean that the norm of the base element in the ith term ($i > 1$) is never greater than the norm of the element in the $(i - 1)$-st terms. Then, among the various possible representations of the element ξ by a linear form there will be one or more representations in which the norm of the element $x_{m\alpha}$ in the leading term of the form is a minimum. For the norm of the leading element $x_{m\alpha}$ in any representation must be at least as great as the norm of ξ itself, otherwise the form ax would represent an element of smaller norm than ξ. We shall call a representation in which the norm of the leading base element is a minimum a *reduced representation.*

Let us now consider an arbitrary sequence of elements ξ_i of Ξ with reduced representations of the following sort,

$$\xi_i = a_{mia_i} x_{mia_i} + \cdots, \tag{12.2}$$

and inquire under what conditions the sequence converges to the zero element. We shall prove the following theorem:

A necessary and sufficient condition that the sequence of elements ξ_i converge to the zero element is that the norms $1/m_i$ of the leading base elements in the right hand members of their reduced representations (12.2) *converge to zero.*

The condition is obviously sufficient since the norm of ξ_i cannot exceed the norm of the leading base element $x_{m_i\alpha_i}$. It is also necessary. For let m be any positive integer. Then if the elements ξ_i converge to zero, all but a finite number of them will lie in the neighborhood of zero determined by the base elements of norms less than $1/m$. Thus, there will exist a positive n such that for i greater than n the norm $1/m_i$ of the leading element of a reduced representation of ξ_i will be smaller than $1/m$. Hence the norms $1/m_i$ must also converge to zero.

Now suppose we change the norm of each element ξ of Ξ so as to make it equal to the norm of the leading base element $x_{m\alpha}$ in a reduced representation of ξ. Then the group Ξ will continue to be an indexed group, for we can readily verify that the new norms satisfy Conditions (i), (ii), and (iii) of §2. Moreover, the change of norm will not affect the topological structure of the group Ξ since the meaning of convergence to zero will not be altered. We shall call the new norms the norms determined by the basis $x_{m\alpha}$ of Ξ.

Every free group Ξ is completely determined (in the topological sense) when we know how many base elements there are of norms $1, \frac{1}{2}, \frac{1}{3}, \cdots$ *respectively in a given free basis of Ξ.*

The theorem is obvious, since we can replace the norms of the elements ξ of Ξ by the norms determined by the given free basis of Ξ without altering the topological structure of Ξ.

13. *Every closed subgroup* H *of a free indexed group Ξ is a free indexed group.*

This theorem is analogous to the well known theorem that every subgroup of an ordinary free group (without indices) is a free group. To prove it we shall assume that the elements $x_{m\alpha}$ of a free basis of the group Φ can be well ordered, though we shall not assume that they are countable. We can then treat the second subscript α appearing in the symbol of a base element $x_{m\alpha}$ as a (finite or transfinite) ordinal. The norm of an element of Ξ will be assumed to be the norm determined by the basis $x_{m\alpha}$, cf. §12.

For convenience, we shall arrange the elements $x_{m\alpha}$ of a free basis of Ξ in a definite linear order by prescribing that an element $x_{m\alpha}$ shall precede an element $x_{n\beta}$:

(α) Whenever the norm $1/m$ of $x_{m\alpha}$ is greater than the norm $1/n$ of $x_{n\beta}$;

(β) Whenever the norm $1/m$ of $x_{m\alpha}$ is the same as the norm $1/n$ of $x_{n\beta}$ but the second subscript α of $x_{m\alpha}$ is a greater ordinal than the second subscript β of $x_{n\beta}$.

A sequence of elements $x_{m\alpha}$ will be said to be in *normal order* if the ordering of the elements in the sequence is consistent with the ordering prescribed above. Every element ξ of Ξ can evidently be expressed as a convergent linear form

$$\xi = ax = a_{m\alpha}x_{m\alpha} + \cdots \tag{13.1}$$

in which the elements $x_{m\alpha}$ appear in their normal order. Let us also observe, once for all, that if the terms of an arbitrary linear from ax can be so arranged that the elements $x_{m\alpha}$ in the various terms follow one another in normal order, then the form ax is necessarily convergent and therefore represents an element ξ of Ξ. For the number of terms corresponding to each value of the first subscript m must be finite, since the values taken on by the second subscript α form a monotonically decreasing (and therefore finite) sequence of ordinals.

We can now obtain a basis of the closed subgroup H of Ξ in the following manner. Let the symbol ξ in relation (13.1) represent any element of the subgroup H. Then, among the elements of H with the same leading element $x_{m\alpha}$ as ξ we shall select any one element

$$y_{m\alpha} = b_{m\alpha}x_{m\alpha} + \cdots \tag{13.2}$$

such that its leading coefficient $b_{m\alpha}$ is positive and numerically as small as possible. The set of all elements $y_{m\alpha}$ corresponding to the various possible choices of the leading element $x_{m\alpha}$ will be the desired basis of the subgroup H, as we shall prove in a moment. We shall assume that the *normal ordering* of the elements $y_{m\alpha}$ is the one determined by the ordering of their respective leading elements $x_{m\alpha}$. Of course, it is not necessary that there be an element $y_{m\alpha}$ corresponding to each element $x_{m\alpha}$.

Since the norm of an element $y_{m\alpha}$ is the same as the norm of its leading element $x_{m\alpha}$ in Representation (13.2), it follows at once that every linear form

$$cy = c_{m\alpha}y_{m\alpha} + \cdots \tag{13.3}$$

in which the elements $y_{m\alpha}$ appear in their normal order is a convergent sum. Moreover, by the lemma of §10, the element cy will be expressible in terms of the base elements $x_{m\alpha}$,

$$cy = c_{m\alpha}b_{m\alpha}x_{m\alpha} + \cdots, \tag{13.4}$$

by substituting Expressions (13.2) directly in (13.3). The leading term of cy, when expressed in terms of the $x_{m\alpha}$'s will be the one indicated in (13.4). Therefore the norm of cy will be the same as the norm of its leading element $y_{m\alpha}$ in Representation (13.3). The elements $y_{m\alpha}$ will thus form a basis of a free subgroup H′ of Ξ consisting of all elements of the type cy.

Now the subgroup H′ is surely contained within the subgroup H since the base elements $y_{m\alpha}$ of H′ are all contained in H and since H is a closed group. Therefore, to complete the argument, we have only to identify the two subgroups H and H′ by showing that every element of the subgroup H may be represented as a convergent linear form cy. Consider any element

$$\eta = d_{m\alpha}x_{m\alpha} + \cdots \tag{13.5}$$

of the subgroup H. Then the leading coefficient $d_{m\alpha}$ of η must be exactly divisible by the leading coefficient $b_{m\alpha}$ of the base element $y_{m\alpha}$ in (13.2). Otherwise there would be a linear combination of the elements η and $y_{m\alpha}$ with a positive

leading coefficient $e_{m\alpha}$ numerically smaller than $b_{m\alpha}$, contrary to the definition of $y_{m\alpha}$. We can thus express the element η as the sum

$$\eta = f_{m\alpha} y_{m\alpha} + \eta_1,$$

where η_1 is expressible as a linear form,

$$\eta_1 = g_{n\beta} x_{n\beta} + \cdots$$

such that its leading element $x_{n\beta}$ follows the leading element $x_{m\alpha}$ of η in the normal ordering of the base elements of Ξ. By operating in a similar way on η_1, we can write

$$\eta_1 = f_{n\beta} y_{n\beta} + \eta_2,$$

whence

$$\eta = f_{m\alpha} y_{m\alpha} + f_{n\beta} y_{n\beta} + \eta_2,$$

and so on. If η_i is the remainder term after the process has been repeated for i steps, then it is evident that η_i converges to zero as i continues to increase. Therefore, in the limit, we shall have the desired representation of η,

$$\eta = f_{m\alpha} y_{m\alpha} + f_{n\beta} y_{n\beta} + \cdots.$$

This completes the argument.

14. *Every complete indexed group Ξ is representable as the difference Φ (mod Ψ) of two free indexed groups Φ and Ψ.*

Proof. Let the elements of any basis of the group Ξ be denoted by $x_{m\alpha}$. Then we can let the group Φ be the free group of all convergent linear forms

$$ax = \Sigma a_{m\alpha} x_{m\alpha} \tag{14.1}$$

in the base elements $x_{m\alpha}$, where two linear forms are regarded as distinct if they are *formally* distinct, whether or not they represent the same element of the group Ξ. Moreover, we can let the group Ψ be the closed free subgroup of Φ consisting of all linear combinations (14.1) that represent the zero element of Ξ. The groups Ξ and Φ (*mod* Ψ) are isomorphic, as may be seen by putting each element ξ of Ξ into correspondence with the co-set made up of all representations cx of ξ. Moreover, the correspondence in question is homeomorphic. For the norm of the element ξ, as determined by the basis $x_{m\alpha}$, is precisely the norm of the leading base element in a reduced representation cx of ξ, (§12), and this, in turn, is precisely the norm of the co-set corresponding to ξ, (§7).

15. **Matrices.** Suppose we have an arbitrary representation of a group Ξ as the difference Φ (*mod* Ψ) of two free indexed groups. Let $x_{n\beta}$ be the elements of a free basis of Φ and $y_{m\alpha}$ the elements of a free basis of Ψ. Then the group $\Xi = \Phi$ (*mod* Ψ) is completely determined when we know the relations

$$y_{m\alpha} = \Sigma a_{m\alpha, n\beta} x_{n\beta} \tag{15.1}$$

expressing the base elements of Ψ in terms of the base elements of Φ. It will be convenient to represent Relations (15.1) in the abridged form

$$y = Ax, \tag{15.2}$$

where A may be thought of as the symbol for an infinite matrix. The *row* $m\alpha$ of the matrix A will consist of all coefficients $a_{m\alpha,n\beta}$ with fixed values m and α of the first two subscripts; the *column* $n\beta$ of A, of all coefficients $a_{m\alpha,n\beta}$ with fixed values n and β of the last two. If we assign a normal linear order to the base elements $x_{n\beta}$ and $y_{m\alpha}$ respectively, (§13), a normal order will be automatically determined for the rows and columns of the matrix A. However, it will seldom be necessary to think of the rows and columns of A as arranged in any particular order. The symbol A_{mn} will be used to designate the *minor* of A consisting of all coefficients $a_{m\alpha,n\beta}$ with fixed values m and n of their first and third subscripts. We can always think of the matrix A as made up of blocks of terms of the form A_{mn}. Thus, to put the matter schematically, we shall have

$$\mathrm{A} = \begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & \cdot \\ A_{21}, & A_{22}, & A_{23}, & \cdot \\ A_{31}, & A_{32}, & A_{33}, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{15.3}$$

A matrix A will be said to be *regular* if, and only if, its representation (15.3) fulfils the following two conditions:

(α) Each row of each minor A_{mn} contains at most a finite number of non vanishing elements.

(β) Each column of minors A_{mn}, (n fixed, $m = 1, 2, 3, \cdots$), contains at most a finite number of minors A_{mn} in which non vanishing elements appear.

If we are assuming that the indexing of the subgroup Ψ of Φ is consistent with the indexing of Φ then the matrix of relations (15.1) will have a representation of the form

$$A = \begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & \cdot \\ 0 \ , & A_{22}, & A_{23}, & \cdot \\ 0 \ , & 0 \ , & A_{33}, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \tag{15.4}$$

with vanishing minors below the main diagonal. It will thus satisfy Condition (β) in the definition of a regular matrix. Moreover, it will also satisfy Condition (α), since (α) is merely the condition that each base element $y_{m\alpha}$ be expressible as a *convergent* linear form in the base elements $x_{n\beta}$. Thus the matrix A of relations (15.1) will be regular. We shall call the matrix A the *matrix* of the representation $\Xi = \Phi \ (mod\ \Psi)$. Equation (15.2) will be the *equation* of the representation.

The real significance of regular matrices will be brought out more clearly when we discuss continuous transformations in §17.

16. **Free Groups.** At this point we shall make a slight digression to determine all possible types of free groups with countable free bases.

(*a*) First, we shall consider the case of a free group Ξ such that the norms of all its base elements $x_{m\alpha}$ exceed some fixed positive number $1/n$. The group Ξ will then be a discrete group, that is to say, a group without limit elements. Therefore we shall not change its topological structure if we redefine the norm of each element of Ξ other than the zero element as 1. The group Ξ will thus reduce either to a group Φ_n with a finite basis composed of n elements x_{1i}, $(i = 1, 2, \cdots, n)$, or to a group Φ_ω with an infinite basis composed of a countable set of elements x_{1i} $(i = 1, 2, \cdots)$.

(*b*) Next, we have the case where the norms of the base elements have the greatest lower bound 0 but where there are never more than a finite number of base elements of the same norm $1/m$. This time, the base elements can be arranged in an infinite sequence with their norms decreasing steadily to zero.[5] Moreover, without affecting the meaning of convergence to 0 (and therefore without altering the topological structure of the group), we can redefine the norms of the base elements so that the norm of the ith one shall be $1/i$. The group Ξ will thus reduce to a group Φ^* with a countable basis of the form x_{i1} $(i = 1, 2, \cdots)$.

(*c*) Thirdly, there is the case where the norms of the base elements have the greatest lower bound zero and where for a finite (but not for an infinite), number of values of m there are infinitely many base elements of the same norm $1/m$. Let the last one of these values of m be μ. Then we can arrange the base elements of Ξ in two infinite sequences, a first sequence composed of all elements of norms greater than or equal to $1/\mu$ and a second sequence composed of the remaining base elements, arranged, if we like, so that their norms decrease steadily to zero. Moreover we can redefine the norms of the base element so that the norm of every element in the first sequence shall be 1 and that the norm of the ith element in the second sequence shall be $1/i$, $(i = 1, 2, 3, \cdots)$. We shall thereby reduce the group Ξ to a group Φ_ω^* with a basis composed of the element x_{11}, the elements x_{1i}, $(i = 2, 3, \cdots)$, and the elements x_{i1}, $(i = 2, 3, \cdots)$.

(*d*) Finally, there is the case where for an infinite number of different values μ_i $(i = 1, 2, 3, \cdots)$ of m the basis of Ξ contains infinitely many elements of the same norm $1/m$. This time we can arrange the base elements into classes c_i such that each class c_i will be composed of all base elements of norms greater than or equal to $1/\mu_i$ but smaller than $1/\mu_{i-1}$. Moreover, we can redefine the norms of the base elements so that the ones in the ith class c_i shall all have the norm $1/i$. Each class c_i will contain a countable infinity of elements; therefore, the group Ξ will reduce to a group Φ with a free basis of the form x_{ij} $(i, j = 1, 2, 3, \cdots)$.

It is still necessary to prove that the various group types Φ_n, Φ_ω, etc. are all distinct. This can easily be done as follows. First, we observe that every free

[5] Actually, it is sufficient that they be arranged in an arbitrary infinite sequence.

group Ξ possesses an invariant sub-group Ξ' consisting of all elements of Ξ of the form $2x$, where x can range over all the elements of Ξ. Therefore the group Ξ also possesses an invariant difference group Ξ (*mod* Ξ'), the elements of which can obviously be represented by all linear combinations of the base elements of Ξ with coefficients reduced modulo 2. Now suppose we form this difference group for each of the groups Φ_n, Φ_ω, $\cdots$. From Φ_n we shall obtain a finite group with 2^n elements, from Φ_ω an infinite discrete group with a countable number of elements, from Φ^* a compact group with limit elements, from Φ^*_ω a group which is not compact but which has a compact neighborhood of the origin, and from Φ a group which is not compact and which has no compact neighborhood of the origin. The group types

$$\Phi_1, \Phi_2, \cdots, \Phi_\omega, \Phi^*, \Phi^*_\omega, \Phi$$

are, therefore, all distinct.

A complete classification of all free groups, whether or not they have a countable basis, can, no doubt, be made without too much difficulty with the aid of transfinite numbers.

17. **Continuous Transformations.** Let us now inquire under what conditions a set of linear relations,

$$x_{m\alpha} = \Sigma a_{m\alpha,n\beta} y_{n\beta}, \tag{17.1}$$

determines a continuous mapping of a group Ξ with the basis $x_{m\alpha}$ on a group H with the basis $y_{n\beta}$. We shall begin with the case where the two groups Ξ and H are free.

A necessary and sufficient condition that a matrix equation

$$x = Ay \tag{17.2}$$

determine a continuous mapping of a free group Ξ with the free base elements $x_{m\alpha}$ on a free group H *with the free base elements $y_{n\beta}$ is that the matrix A be regular,* (§15).

We are here regarding Relation (17.2) as an abridged way of writing Relation (17.1).

First we prove that if Relation (17.2) determines a continuous mapping of Ξ on H then the matrix A must be regular. Condition (α) for a regular matrix must evidently be satisfied, since it is merely the condition that each base element $x_{m\alpha}$ be representable as a convergent linear form in the base elements $y_{n\beta}$. Condition (β) must also be satisfied. For if there were any column of minors in the representation (15.3) of A containing an infinite sequence of non vanishing minors

$$A_{mn}, \qquad (n \text{ fixed}; m = 1, 2, 3, \cdots)$$

it would mean that we could select an infinite sequence of base elements $x_{mi\alpha i}$ such that the image of each element $x_{mi\alpha i}$ would contain a term in a base element $y_{n\beta i}$ of norm $1/n$. Thus the sequence of elements $x_{mi\alpha i}$ would converge to zero, whereas the image of the sequence would not. The transformation $x = Ay$ would therefore fail to be continuous.

Conversely, if the matrix A is regular then Relation (17.2) determines a continous mapping. For if Condition (α) is satisfied then every base element $x_{m\alpha}$ will be expressible as a convergent linear form in the base elements $y_{n\beta}$. Moreover if Condition (β) is satisfied, then, by the lemma of §10, every element cx of Ξ will also be expressible as a convergent linear form in the base elements $y_{n\beta}$,

$$cx = cAy.$$

Relation (17.2) therefore determines a mapping. Moreover, the mapping is continuous. For suppose we take any infinite sequence of elements $c^{(i)}x$ converging to zero. Then the images $c^{(i)}Ay$ of the elements $c^{(i)}x$ will also have to converge to zero. For, by Condition (β), if ν is any positive integer there exists a positive integer $\mu(\nu)$ such that all the minors after the μth in the first ν columns of (15.3) are zero,

$$A_{mn} = 0, \qquad (m > \mu,\ n \leq \nu').$$

Therefore, since all but a finite number of terms of the sequence $c^{(i)}$ will have norms smaller than $1/\mu$ all but a finite number of them will have images of norms smaller than $1/\nu$. Thus the images of the elements $c^{(i)}x$ will also converge to zero. The transformation (17.2) is therefore continuous.

18. *The resultant of two continuous transformations is a continuous transformation.*

For if $\lim x_i = x$ implies $\lim y_i = y$ and $\lim y_i = y$ implies $\lim z_i = z$, then, of course, $\lim x_i = x$ implies $\lim z_i = z$.

The product AB of two regular matrices A and B is a regular matrix.

By the product we mean, of course, a matrix with the elements

$$c_{m\alpha,n\beta} = \sum_{s\gamma} a_{m\alpha,s\gamma} b_{s\gamma,n\beta}, \tag{18.1}$$

assuming that such a matrix exists.

The theorem we are proving is a little more than a corollary of the previous one. The previous theorem tells us that the resultant of the transformations determined by the regular matrices A and B is a transformation with a regular matrix C. However, it does not tell us that we have $C = AB$ or even that AB has any meaning. We shall, therefore, write out explicitly the transformations determined by A and B,

$$x_{m\alpha} = \Sigma a_{m\alpha,s\gamma} z_{s\gamma}, \qquad z_{s\gamma} = \Sigma b_{s\gamma,n\beta} y_{n\beta}.$$

Now, since the matrices A and B are regular, it follows immediately by the lemma of §10 that we can express each element $x_{m\alpha}$ in terms of the elements $y_{n\beta}$ by an equation of the following sort:

$$x_{m\alpha} = \sum_{\substack{s\gamma \\ n\beta}} (a_{m\alpha,s\gamma} b_{s\gamma,n\beta}) y_{n\beta}. \tag{18.2}$$

But equations (18.2) can be none other than the equations of the resultant of the transformations A and B. Their matrix must, therefore, be the matrix C. We have thus identified the matrix C with the product AB.

19. The theorem of §17 no longer holds if the groups Ξ and H are not free groups. We can, however, prove the following:

Every continuous mapping of a complete group Ξ on a complete group H *can be represented by a relation of the form*

$$x = Ay, \tag{19.1}$$

where A is a regular matrix.

We have only to express each base element $x_{m\alpha}$ of Ξ as a *reduced* representation, §12, in the base elements of H to obtain Relation (19.1). The regularity of the matrix A follows, at once, by the argument of §17. There may very well be other ways of representing the base elements $x_{m\alpha}$ which will not lead to a regular matrix A.

20. **Homeomorphic Transformations.** A transformation $y = Bx$ will be called the *inverse* of a transformation $x = Ay$ provided the resultant of the two transformations is the identical transformation *regardless of the order in which the transformations are made.*

If $x = ABx$ is the identical transformation it does not necessarily follow that $y = BAy$ is also the identical transformation, as may be seen by letting A be the matrix of the transformation

$$x_{1i} = y_{1(i+1)}, \qquad (i = 1, 2, 3, \cdots), \tag{20.1}$$

and B the matrix of the transformation

$$y_{11} = 0, \qquad y_{1(i+1)} = x_{1i}, \qquad (i = 1, 2, 3, \cdots). \tag{20.2}$$

The resultant of (20.1) followed by (20.2) is then the identical transformation, but the resultant of (20.2) followed by (20.1) is the degenerate transformation

$$y_{11} = 0, \qquad y_{1(i+1)} = y_{1(i+1)}, \qquad (i = 1, 2, 3, \cdots).$$

A transformation $x = Ay$ will be said to be *homeomorphic* if it establishes a homeomorphic correspondence between the elements cx of a free group Ξ and the elements dy of a free group H.

A necessary and sufficient condition that a transformation $x = Ay$ be homeomorphic is that it be continuous and have a continuous inverse $y = Bx$. (This theorem is obvious.)

The matrix of a homeomorphic transformation will be said to be *intact.* An intact matrix A will thus be any regular matrix possessing a regular inverse B, i.e., where B is such that

$$AB = 1, \qquad BA = 1.$$

Two regular matrices M and N will be said to be equivalent if we have

$$N = AMB,$$

where A and B are intact.[6]

A relation $x' = Ax$, where A is intact, may, of course, be thought of as determining a change from a basis x of a free group Ξ to a different basis x'.

21. It is necessary to call attention at this point to a special type of homeomorphic transformation which will be needed in the next section. There is first a matter of notation to be attended to. If we have an arbitrary transformation

$$x' = Ax, \tag{21.1}$$

it will sometimes be convenient to split up the elements of each of the bases x and x' into two sets and to write the transformation (21.1) in the form

$$\begin{aligned} y' &= Py + Qz, \\ z' &= Ry + Sz. \end{aligned} \tag{21.2}$$

The matrix A will then be represented schematically in terms of its minors P, Q, R, and S as follows:

$$A = \left| \begin{matrix} P, & Q \\ R, & S \end{matrix} \right|.$$

In this notation, the minors A_{mn} of A, §15, will be

$$A_{mn} = \left| \begin{matrix} P_{mn}, & Q_{mn} \\ R_{mn}, & S_{mn} \end{matrix} \right|,$$

from which we see, at once, that:

A necessary and sufficient condition that the matrix A be regular is that all four of its minors P, Q, R, and S be regular.

The desired homeomorphic transformations will be transformations $x' = Ax$ with matrices of the form

$$A = \left| \begin{matrix} 1, & P \\ 0, & 1 \end{matrix} \right|,$$

where P is any regular matrix. Evidently *the matrix A is intact.* For it is regular, and we see by inspection that it has the regular inverse

$$A = \left| \begin{matrix} 1, & -P \\ 0, & 1 \end{matrix} \right|.$$

The transformation $x' = Ax$ is therefore homeomorphic. By a similar argument it may be shown that the matrix

$$A = \left| \begin{matrix} 1, & 0 \\ P, & 1 \end{matrix} \right|$$

[6] This definition does not quite correspond to the one used in Alexander and Cohen, loc. cit.

is intact. The rôles of the two sets of base elements have merely been interchanged.

One further remark. Let A and A' be the respective matrices of certain representations of two groups Ξ and Ξ', §15. Then the matrix

$$\begin{vmatrix} A, 0 \\ 0, A' \end{vmatrix}$$

is obviously the matrix of a representation of the direct sum $\Xi \oplus \Xi'$ of Ξ and Ξ'.

22. **A Criterion for Homeomorphism.** We shall now consider two arbitrary complete groups,

$$\Xi = \Phi \ (mod\ \Psi) \tag{22.1}$$

and

$$\Xi' = \Phi' \ (mod\ \Psi'), \tag{22.2}$$

and ask ourselves under what conditions the groups are homeomorphic. Let us suppose that the equations of the two representations (22.1) and (22.2) are

$$y = Ax \text{ and } y' = A'x' \tag{22.3}$$

respectively. Then if the matrices A and A' are equivalent,

$$A' = EAF, \qquad (E, F \text{ intact}),$$

the groups Ξ and Ξ' are obviously homeomorphic. For if we change the basis of Φ according to $x = Fx'$ and the basis of Ψ according to $y = E^{-1}y'$ we shall reduce the equation of the first representation to the equation of the second. The groups Ξ and Ξ' may, however, be homeomorphic even when the matrices A and A' are not equivalent, as the following illustration will show. Let Φ be a discrete free group with the base elements x_{1i}, $(i = 1, 2, \cdots)$, all of norm 1, and let Φ' be a similar group with the base elements x_{1i}, $(i = 1, 2, 3, \cdots)$. Moreover, let the groups Φ $(mod\ \Psi)$ and Φ' $(mod\ \Psi')$ be the groups determined in the first case by the equations

$$y_{1i} = 2x_{1i}, \qquad (i = 1, 2, 3, \cdots), \tag{22.4}$$

and in the second case by the equations

$$\begin{aligned} y_{1i} &= 2x_{1i}, && (i \text{ even}), \\ y_{1i} &= x_{1i}, && (i \text{ odd}). \end{aligned} \tag{22.5}$$

The two groups Φ $(mod\ \Psi)$ and Φ' $(mod\ \Psi')$ are then obviously homeomorphic. However, the matrix of equations (22.4) is not equivalent to the matrix of equations (22.5). For if these two matrices were equivalent they would be equivalent, *a fortiori*, if we reduced their coefficients modulo 2. But when we reduce coefficients modulo 2 the first matrix vanishes, whereas the second one does not. Therefore, the second one cannot be the transform of the first one.

To return to the two general groups Ξ and Ξ', let us assume, as we may without essential loss of generality, that the groups Φ and Φ' in Representations (22.1) and (22.2) have no element in common. Then we can at once obtain a new representation of the group Ξ of the form

$$\Xi = [\Phi \oplus \Phi'] \ (mod [\Psi \oplus \Phi']) \tag{22.6}$$

by merely adding the group Φ' to each of the groups Φ and Ψ. The fact that (22.6) actually is a new representation of Ξ is obvious, for the co-sets of the new difference group in (22.6) are merely enlargements of the co-sets of the old one in (22.1). Similarly we can obtain a new representation

$$\Xi' = [\Phi \oplus \Phi'] \ (mod \ [\Phi \oplus \Psi']) \tag{22.7}$$

of Ξ'. The matrices of Representations (22.6) and (22.7) will obviously be of the form

$$B = \begin{vmatrix} A, 0 \\ 0, 1 \end{vmatrix} \text{ and } B' \begin{vmatrix} 1, 0 \\ 0, A' \end{vmatrix} \tag{22.8}$$

respectively.

We shall now prove the following theorem:

A necessary and sufficient condition that the groups Ξ and Ξ' represented by (22.1) *and* (22.2) *be homeomorphic is that the matrices B and B' of their modified representations* (22.6) *and* (22.7) *be equivalent.*

Proof. The equivalence of the matrices B and B' is a sufficient condition for the homeomorphism of the two groups for the same reason that the equivalence of A and A' is sufficient. To show that the condition is also necessary we shall assume that the groups Ξ and Ξ' are homeomorphic and shall prove that, in consequence, the matrices B and B' are equivalent.

Let us go back, for a moment, to Representation (22.1) and (22.2). Since Ξ and Ξ' are homeomorphic there will be two matrix equations

$$x = Kx', \qquad x' = K'x, \qquad (K, K' \text{ regular}), \tag{22.9}$$

the first determining a homeomorphic mapping of Ξ on Ξ', the second determining the inverse mapping of Ξ' on Ξ. Of course, we cannot conclude that the matrix K' is the inverse of the matrix K, because the bases x and x' of Ξ and Ξ' are not, in general, free bases.

Now, in view of Relations (22.9), we shall not be restricted to representing an element of Ξ as a linear form cx in the base x. It will also be possible to represent such an element as a linear form $c'x'$ in the base x', or, if we prefer, as a linear form $ax + a'x'$ in the combined bases x and x', $(ax + a'x' = ax + a'K'x)$. This last representation is the one we shall now consider. It corresponds to representing the group Ξ as a difference group of the form

$$\Xi^* = [\Phi \oplus \Phi'] \ (mod \ \Psi^*), \tag{22.10}$$

where the group Ψ^* consists of all elements $ax + a'x'$ of $\Phi \oplus \Phi^*$ that represent

the zero element of Ξ^*. There are two obvious ways of finding a free basis of the group Ψ^*:

(α) By combining the base elements $y = Ax$ of the group Ψ with a new set of base elements of the form $z = x' - K'x$, where K' is the matrix occurring in (22.9). If we form the basis of Ψ^* in this way, the matrix of the representation (22.10) will be

$$C = \begin{vmatrix} A, & 0 \\ -K', & 1 \end{vmatrix}.$$

(β) By combining the base elements $y' = A'x'$ of the group Ψ' with a new set of base elements $z' = x - Kx'$. The matrix of (22.10) will then be

$$C' = \begin{vmatrix} 1, & -K \\ 0, & A' \end{vmatrix}.$$

The matrices C and C' will, of course, be equivalent, since we can transform one into the other by a proper change of basis of Ψ^*.

The proof may now be completed at once. The matrix B is equivalent to the matrix C. For we have

$$\begin{vmatrix} A, & 0 \\ -K, & 1 \end{vmatrix} = \begin{vmatrix} A, & 0 \\ 0, & 1 \end{vmatrix} \begin{vmatrix} 1, & 0 \\ -K, & 1 \end{vmatrix}, \tag{22.11}$$

as may readily be verified. Moreover, by §21, the last matrix on the right is intact and, therefore, merely corresponds to a change of basis of $\Phi \oplus \Phi'$. By a similar argument, the matrix B' is equivalent to the matrix C'. But the matrices C and C' are equivalent, therefore the matrices B and B' must be equivalent. This is what we set out to prove.

23. **Two Special Cases.** Let Ξ be a free group with a free basis x. Then an equation of the form

$$x' = Ax, \qquad (A \text{ intact}), \tag{23.1}$$

may be thought of either as defining a change of basis of Ξ or as defining a bi-continuous automorphic transformation of the group Ξ into itself. Two special cases are important enough to be worth considering by themselves.

(a) The case where the group Ξ is a free discrete group of the type of Φ_ω, §16, with a countably infinite basis x_{1i}, $(i = 1, 2, 3, \cdots)$. In this case the matrix A has only a single minor A_{mn}, namely, the minor $A_{11} = A$. The matrix A is thus an ordinary infinite matrix of the form

$$A = \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \cdot \\ a_{21}, & a_{22}, & a_{23}, & \cdot \\ a_{31}, & a_{32}, & a_{33}, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{23.2}$$

Moreover, the condition that such a matrix be regular is that there be at most a finite number of non vanishing elements in each of its rows.

(*b*) The case where the group Ξ is a free group of the type of Φ^*, §16, with a countably infinite basis x_{i1}, $(i = 1, 2, 3, \cdots)$. In this case, each of the minors A_{mn} of A contains a single element. The matrix A is thus, again, an infinite matrix of the form (23.2). This time, however, there may be infinitely many non vanishing elements in a row of A, but there can be at most a finite number of non vanishing elements in each column of A if A is to be regular.

Thus, it follows at once that:

The group of all bi-continuous automorphic transformations of the free group Φ_ω into itself is the dual of the group of all bi-continuous transformations of the free group Φ^* into itself.

For each transformation of the first group will be determined by an intact matrix A with "finite rows." Corresponding to this transformation there will be a transformation of the second group determined by a matrix A' with "finite columns," where A' is the transposed matrix of A, $(a'_{ij} = a_{ji})$. The matrix A' will evidently be intact if, and only if, the matrix A is intact. Corresponding to a transformation with the matrix AB will be a dual transformation with the matrix $B'A'$.

24. The remarks in the last section have a bearing on the classification of the homology groups of compact spaces. In our joint paper,[1] Professor Cohen and I showed that every homology group can be resolved into the sum of a free group and a set of groups T_π with matrices of the form

$$B = \begin{vmatrix} \pi, & a_{12}, & a_{13}, & \cdot \\ 0, & \pi, & a_{23}, & \cdot \\ 0, & 0, & \pi, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \tag{24.1}$$

with zeros below the main diagonal, the same prime number π along the main diagonal, and arbitrary integers above. However, we failed to classify these last groups correctly. Now we see that each group T_π with a matrix B corresponds to a "dual" discrete group T'_π such that the matrix of T'_π is the transpose B' of the matrix B of T_π. It follows at once, by §§23 and 24, that:

A necessary and sufficient condition for the equivalence of two groups T_π and $\bar{T}_\pi$ with matrices of the form (24.1) *is that the two discrete groups T'_π and $\bar{T}'_\pi$ dual to T_π and $\bar{T}_\pi$ respectively, be equivalent.*

Necessary and sufficient conditions for the equivalence of two groups T'_π and $\bar{T}'_\pi$ have been given by Ulm,[2] so that the classification of the homology groups of compact spaces may now be regarded as complete.

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[2] Loc. cit.

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ANALYTIC FUNCTIONS WITH ALMOST PERIODIC COEFFICIENTS

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1. Introduction

In this paper we shall discuss the analytic functions defined by power series of the form

$$G_1(z) = \sum_0^\infty f(n)\, z^n,$$

where $f(t)$ is a uniform almost periodic function of t. Discarding the case $f(n) = 0$, the radius of convergence of the power series is unity. When $f(t)$ reduces to an exponential polynomial,

$$f(t) = \sum_1^N a_p e^{-i\lambda_p t},$$

$G_1(z)$ is a rational function,

$$G_1(z) = \sum_1^N a_p \frac{1}{1 - ze^{-i\lambda_p}};$$

all its poles are simple and lie on the unit circle. In the general case we obtain a formal infinite series not necessarily convergent.

Now our aim is precisely to show that the analytic character of the function is exactly the same as if this infinite series were convergent. Our methods consist of replacing the "multiplier" $(1 - ze^{-i\lambda_p})$, by a more suitable function of the same kind, different from the former one only in intervals of λ containing no exponents of $f(t)$. Such replacements of multipliers have been used previously and proved to be very useful tools in dealing with Fourier series and integrals.[2] This problem about power series suggests an analogous one for Dirichlet series

$$\sum_1^\infty f(\log n)\, 1/n^s$$

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[1] Presented to the American Mathematical Society, December 2, 1933.

[2] Cf. S. Bochner, *Vorlesungen über Fouriersche Integrale*. §23 and §32; also S. Bochner, *Über gewisse Differential und allgemeinere Gleichungen, deren Lösungen fastperiodisch sind.* I, II and III. Math. Ann. **102** (1929) pp. 489–504, **103** (1930), pp. 588–597, **104** (1931), pp. 579–587. N. Wiener, *The operational calculus*. Math. Ann. **95** (1925–26), pp. 557–584.

and for Laplace integrals

$$\int_0^\infty e^{-st} f(t)\, dt.$$

The results are similar in all cases; we shall begin with and treat carefully the case of the Laplace integral, since it is free from complications arising in the two other cases.

2. The Laplace integral

The u.a.p. function $f(t) \sim \sum_1^\infty a_p e^{i\lambda_p t}$ being continuous and bounded, the integral

$$G_1(s) = \int_0^\infty e^{-st} f(t)\, dt$$

defines in the half plane $\sigma = \Re(s) > 0$ an analytic function $G_1(s)$. The kernel

$$K_1(s, t) = \begin{cases} e^{-st}, & t > 0, \\ 0, & t < 0, \end{cases} \qquad \sigma > 0$$

can be represented by the integral

$$(1) \qquad K_1(s, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\lambda t}}{s - i\lambda}\, d\lambda.^3$$

With this kernel it is essential to suppose $\sigma \neq 0$; so that in order to study the analytic behavior of $G_1(s)$ on the line $\sigma = 0$, and in particular in the neighborhood of the point $s = 0$, a new kernel must be introduced:

$$(2) \qquad K_2(s, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\lambda t}}{s - \varphi(\lambda)}\, d\lambda,$$

where the complex valued function $\varphi(\lambda)$ of λ is supposed to satisfy the conditions:

(a) $\varphi(\lambda)$ possesses two continuous derivatives,

(b) $|\varphi(\lambda)| > 1 - \delta$ for a certain δ, $0 < \delta < \frac{1}{4}$,

(c) $\varphi(\lambda) = i\lambda$ for $|\lambda| \geqq 1$.

Such functions will obviously exist for any given δ. Furthermore, let $\mathfrak{C}_\delta$ denote the interior of the circle $|s| < 1 - 2\delta$. When $s \subset \mathfrak{C}_\delta$, $|s - \varphi(\lambda)| > \delta$; the kernel $K_2(s, t)$ exists (for $t = 0$ as a principal value) and is absolutely integrable for all s. To prove these statements it is simplest to compare $K_2(s, t)$ with the original kernel $K_1(s, t)$. Let s_0 be any fixed point in $\mathfrak{C}_\delta$, whose real part is positive. We have

$$K_2(s, t) = \{K_2(s, t) - K_2(s_0, t)\} + \{K_2(s_0, t) - K_1(s_0, t)\} + K_1(s_0, t),$$

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[3] Cf., for example, Bochner, *Vorlesungen über Fouriersche Integrale*, p. 52. For $t = 0$. the integral has to be defined as a principal value.

where each term can be evaluated separately. The first one, for example, takes the form

$$K_2(s,t) - K_2(s_0,t) = (s_0 - s)\int_{-\infty}^{+\infty} \frac{e^{-i\lambda t}}{(s-\varphi(\lambda))(s_0-\varphi(\lambda))}\,d\lambda.$$

When $t \neq 0$, two integrations by parts allow us to write

$$= \frac{s-s_0}{t^2}\int_{-\infty}^{+\infty} e^{-i\lambda t}\frac{\partial^2}{\partial\lambda^2}\left\{\frac{1}{(s-\varphi(\lambda))(s_0-\varphi(\lambda))}\right\}d\lambda.$$

In both forms we are now dealing with absolutely convergent integrals and it is easily verified that this first term is everywhere continuous in t and absolutely $\leqq$ Min $(C_1, C_2 t^{-2})$, where C_1 and C_2 are absolute constants (independent of s in $\mathfrak{C}_\delta$). Similar results hold true for the second term, while the third one given by (1) is completely known.

Furthermore, the inversion formulae for Fourier integrals[4] allow us to invert equations (2). We obtain

$$\frac{1}{s-\varphi(\lambda)} = \int_{-\infty}^{+\infty} K_2(s,t)\,e^{i\lambda t}\,dt \tag{3}$$

for any λ and any $s \subset \mathfrak{C}_\delta$.

Substituting the new kernel for the original one in (1) we define on $\mathfrak{C}_\delta$ a new function

$$G_2(s) = \int_{-\infty}^{+\infty} K_2(s,t)\,f(t)\,dt$$

and finally in the half plane $\sigma < 0$ a third function

$$G_3(s) = -\int_0^{\infty} e^{st}\,f(-t)\,dt,^5$$

the function $G_3(s)$ evidently being analytical in its domain of definition.

In the special case where $f(t)$ reduces to an exponential polynomial, say $f(t) = \sum_1^N a_p e^{i\lambda_p t}$, the functions $G_i(s)$ can be computed:

$$G_1(s) = \sum_1^N a_p \frac{1}{s-i\lambda_p},$$

$$G_2(s) = \sum_1^N a_p \int_{-\infty}^{+\infty} K_2(s,t)\,e^{i\lambda_p t}\,dt = \sum_1^N a_p \frac{1}{s-\varphi(\lambda_p)}$$

by equation (3), and finally,

$$G_3(s) = \sum_1^N a_p \frac{1}{s-i\lambda}.$$

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[4] Cf., for example, Bochner, loc. cit., p. 42.

[5] The kernel $-e^{st}$ is the old kernel $K_1(s,t)$ for negative σ.

All three functions are rational and $G_1 = G_3$. If, moreover, all λ_p are $\geqq 1$ in absolute value, $\varphi(\lambda_p)$ will be equal to $i\lambda_p$ and all three $G(s)$ will coincide. The different integrals are representations of the same function, valid in different regions.

The use of the rather deep inversion formulae can be avoided entirely. The difference

$$G_2(s) - G_1(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, f(t) \int_{-1}^{+1} \frac{(\varphi(\lambda) - i\lambda)\, e^{-i\lambda t}}{(s - i\lambda)\,(s - \varphi(\lambda))}\, d\lambda$$

$$= O\left(\frac{1}{N}\right) + \frac{1}{2\pi}\int_{-N}^{+N}\int_{-1}^{+1}.$$

For $f(t) = e^{i\lambda_0 t}$, $|\lambda_0| \geqq 1$ we have

$$|G_2 - G_1| \leqq O\left(\frac{1}{N}\right) + \frac{1}{2\pi}\left|\int_{-1}^{+1}\int_{-N}^{+N}\right|$$

$$= O\left(\frac{1}{N}\right) + \frac{1}{2\pi}\left|\int_{-1}^{+1} \frac{(\varphi(\lambda) - i\lambda)}{(s - i\lambda)\,(s - \varphi(\lambda))} \cdot \frac{e^{iN(\lambda - \lambda_0)} - e^{-iN(\lambda - \lambda_0)}}{i\,(\lambda - \lambda_0)}\, d\lambda\right|$$

$$= O\left(\frac{1}{N}\right).$$

Hence $G_2 = G_1$, for all values s for which both functions are defined.

Consider now a sequence $f_n(t)$ of u.a.p. functions approaching uniformly the limiting function $f(t)$. For each n we can form the corresponding $G_i^{(n)}(s)$. For each s (in the respective regions where the different $G_i(s)$ are defined)

$$G_i(s) = \lim_{n\to\infty} G_i^{(n)}(s), \qquad (i = 1, 2, 3)$$

the limit being uniform in every closed, finite domain. For G_2, for example, as soon as $|f(t) - f_n(t)| < \epsilon$ we have for any s in $\mathfrak{C}_3$

$$|G_2^{(n)}(s) - G_2(s)| \leqq \epsilon \int_{-\infty}^{+\infty} |K_2(s, t)|\, dt \leqq M\epsilon,$$

where M is independent of the position of s. Similar arguments hold for G_1 and G_3.

These last two remarks enable us to treat now the general case $f(t) \sim \sum_1^\infty a_p e^{i\lambda_p t}$. There exist for such a function a sequence of exponential polynomials $f_n(t)$ converging uniformly to $f(t)$ such that for each n all the exponents of $f_n(t)$ are taken among the exponents of $f(t)$. Since $G_2(s)$ is the uniform limit of the $G_2^{(n)}$, which are analytical in $\mathfrak{C}_3$, G_2 itself will be analytical.[6]

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[6] The analiticity of G_2 could of course have been proved at the beginning by direct computation of the derivative. We avoided doing so since we obtain it now as a simple consequence of our results.

Supposing all exponents of $f(t)$ to be $\geqq 1$ in absolute value, the same will be true for the exponents of the polynomials. Therefore at each step $G_1^{(n)} = G_2^{(n)} = G_3^{(n)}$ and these functions converge uniformly to G_1, G_2, G_3. But the circle $\mathfrak{C}_\delta$ overlaps with the regions where G_1 and where G_2 are defined. Furthermore, taking account of the fact that the number δ is arbitrarily small, we see that $G_1(s)$, defined originally only in $\sigma > 0$, remains regular on the segment $\sigma = 0$, $|\tau| < 1$; its analytic continuation over this segment leads to the function $G_3(s)$ in $\sigma < 0$. A dilation and translation of the s-plane in the direction of the τ-axis allows us to generalize this result to

THEOREM 1. *If no exponent of the u.a.p. function $f(t)$ lies in the open interval $\Lambda_1 < \lambda < \Lambda_2$, the function*

$$G_1(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-st}\, dt,$$

defined and analytical in $\sigma > 0$, remains regular on the segment $\sigma = 0$, $\Lambda_1 < \tau < \Lambda_2$. If several gaps occur among the exponents, $G_1(s)$ can be extended over each one, the continuation leading always to the same function

$$G_3(s) = -\int_{\infty}^{+\infty} e^{st} f(-t)\, dt.$$

Denoting by $\mathfrak{M}$ the closure of the set $\{i\lambda_p\}$ in the s-plane, we can say that $G(s)$ remains regular on the complementary set. Is it singular for all points of $\mathfrak{M}$? An affirmative answer to this question will be given by considering the growths of $G_1(s)$ and $G_3(s)$ resulting when s approaches a point on the imaginary axis along a parallel to the real axis.

THEOREM 2. *Let $a(\tau) = a_p$ for $\tau = \lambda_p$ (λ_p being the exponents and a_p the corresponding coefficients of $f(t)$) and 0 elsewhere. Then*

$$\lim_{\sigma \to 0,\ \sigma > 0} \sigma\, G_1(\sigma + i\tau) = \lim_{\sigma \to 0,\ \sigma < 0} \sigma\, G_3(\sigma + i\tau) = a(\tau).$$

For the proof we restrict ourselves to $G_1(s)$. Given an arbitrary $\epsilon > 0$, we can find a Fejér polynomial $f_n(t) = \sum_1^N a_p^{(n)} e^{i\lambda_p t}$ such that the absolute value of $f(t) - f_n(t)$ will be $< \epsilon$. The function $G_1(s)$ equals

$$\sum_1^N a_p^{(n)} \frac{1}{s - i\lambda_p} + \int_0^\infty (f(t) - f_n(t))\, e^{-st}\, dt,$$

therefore

$$|\, \sigma\, G_1(\sigma + i\tau) - a(\tau)\,| \leqq \left| \sum_1^N a_p^{(n)} \frac{\sigma}{s - i\lambda_p} - a(\tau) \right| + \epsilon.$$

When $\sigma \to 0$,

$$\overline{\lim}\, |\, \sigma\, G_1(\sigma + i\tau) - a(\tau)\,| \leqq |\, a^{(n)}(\tau) - a(\tau)\,| + \epsilon < 2\epsilon, \quad n < n(\epsilon).$$

This proves theorem 2 since the left hand side does not depend on n or ϵ.

For the points $i\lambda_p$ both G_1 and G_3 tend to infinity. These points, and hence their closure, i.e., all the points of the set $\mathfrak{M}$, are singular points of these two functions. If one exponent, say λ_0, is isolated among the set of all exponents λ_p,

$$f(t) = a_0 e^{i\lambda_0 t} + f_1(t).$$

The G-function generated by $f_1(t)$ is regular at $i\lambda_0$ by theorem 1; the point $i\lambda_0$ is therefore a simple pole for G. Conversely, if G has an isolated singularity at $i\tau_0$, τ_0 must be an isolated exponent of $f(t)$, since, in a neighborhood of τ_0, $a(\tau) = 0$ for $\tau \neq \tau_0$.

THEOREM 3. *All the points of the set $\mathfrak{M}$, the closure of the set $\{i\lambda_p\}$, are singular points for $G_1(s)$ and $G_3(s)$. Isolated singularities are simple poles and correspond to the isolated exponents of $f(t)$.*

Theorem 1 and 3 give a complete analysis of the continuation of $G_1(s)$, in particular the resulting analytic function is uniform.

3. Power series

In this paragraph we study the problem of the continuation of the analytic function defined by the power series

$$G_1(z) = \sum_1^\infty f(n) z^n, \tag{4}$$

where $f(t)$ is again a u.a.p. function with the exponents $-\lambda_p$ and the coefficients a_p.[7] Excluding the case $f(n) = 0$ for all integers, the radius of convergence of (4) is unity. In order to find the behavior of $G(z)$ on the unit circle, in particular at $z = 1$, we introduce here the kernel

$$g_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{in\lambda}}{1 - z\varphi(\lambda)}\, d\lambda, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

where the auxiliary function $\varphi(\lambda)$ shall satisfy the conditions:

(a) $\varphi(\lambda)$ has two continuous derivatives,

(b) $\varphi(\lambda) = e^{-i\lambda}$ when $|e^{i\lambda} - 1| \geqq \rho$,

(c) $\varphi(\lambda)$ is periodic with the period 2π,

(d) $|1 - z\varphi(\lambda)| \geqq \delta$ when z is in $\mathfrak{C}_\delta$: $|z - 1| < \rho - 2\delta$.

For $n \neq 0$ two integrations by parts show that $\sum_{-\infty}^{+\infty} |g_n(z)|$ is uniformly convergent in $\mathfrak{C}_\delta$. We can therefore define in this region a function $G_2(z)$:

$$G_2(z) = \sum_{-\infty}^{+\infty} f(n)\, g_n(z)$$

[7] It is more convenient here to write $f(t) \sim \Sigma a_p e^{-i\lambda_p t}$.

and for $|z| > 1$ finally a third function

$$G_3(z) = -\sum_1^\infty f(-n)\, z^{-n}.$$

When $f(t)$ reduces to an exponential polynomial $\sum_1^N a_p e^{-i\lambda_p t}$ we have

$$G_1(z) = G_3(z) = \sum_1^N \frac{a_p}{1 - z\, e^{-i\lambda_p}}$$

and

$$G_2(z) = \sum_1^N a_p \frac{1}{1 - z\varphi(\lambda_p)}.$$

In case all λ_p are such that $|e^{-i\lambda_p} - 1| \geqq \rho$, all three functions will again coincide. Taking a sequence of Fejér polynomials, we see first that $G_2(z)$ is analytical in $\mathfrak{C}_\delta$, and secondly when all λ satisfy the above condition, that $G_1(z)$ can be continued over the arc φ, $|e^{i\varphi} - 1| < \rho$, to the function $G_3(z)$. A rotation of the z-plane around the origin generalizes this result to any arc of the unit circle which is free from exponents of $f(t)$.

The converse problem, the verification that the continuations of $G_1(z)$ and $G_3(z)$ are possible only over such arcs, is in the case of power series slightly more involved, due to the fact that the same point on the unit circle can be covered by several exponents of $f(t)$. To study the behavior of G_1 and G_3 as z approaches a point $e^{i\varphi_0}$ radially, we pick out from the Fourier series of $f(t)$ all terms whose exponents are of the form $\varphi_0 + 2k\pi$, where k is any integer. These terms form the Fourier series of a certain u.a.p. function which shall be denoted by $f_{\varphi_0}(t)$. Approximating f and f_{φ_0} by Fejér polynomials we prove

$$\lim_{r \to 1} (1 - r)\, G_1(re^{i\varphi_0}) = f_{\varphi_0}(0),^8$$

and a similar result holds for G_3. If $f_{\varphi_0}(0) \neq 0$, the corresponding point is singular, but if it happens to vanish nothing can be concluded. To proceed further we make use of the evident fact that two u.a.p. functions which interpolate the same sequence a_n (i.e., $f(n) = f^*(n) = a_n$) will define the same power series.

LEMMA. *Any given almost periodic sequence a_n can be interpolated by a u.a.p. function in such a way such that for all φ either $f_\varphi(t) \equiv 0$ or $f_\varphi(0) \neq 0$.*

Proof: Let us interpolate the sequence a_n by joining a_n and a_{n+1} by a linear function. For every φ,

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[8] The existence and the value of this limit have already been obtained by A. Walther, *Fastperiodische Folgen und Potenzreihen*, Abh. Hamburg. Sem. 6 (1928), pp. 217–233.

$$e^{i\varphi t} f_\varphi(t) = \lim_{n\to\infty} 1/n \sum_1^n f(t+\nu)e^{i\varphi(\nu+t)},^9$$

and therefore

$$f_\varphi(t) = \lim_{n\to\infty} 1/n \sum_1^n f(t+\nu)e^{i\varphi\nu}.$$

Hence for each φ, $f_\varphi(t)$ will have the same property as $f(t)$ of being linear in any open interval $n < t < n+1$. The function $e^{i\varphi t} f_\varphi(t)$ has the period 1, and in particular, $f_\varphi(n) = f_\varphi(0) \cdot e^{-i\varphi n}$. If $f_\varphi(0)$ vanishes, all $f_\varphi(n)$ will also be zero. But this fact, together with the linearity, obviously implies $f_\varphi(t) \equiv 0$.

By the special interpolation we have avoided all superfluous exponents and every point $e^{i\lambda_p}$ is certainly singular.

THEOREM 4. *Given any almost periodic sequence a_n, the power series $\sum_0^\infty a_n z^n$ defines in the unit circle a function $G_1(z)$ which, if continued analytically over any regular point on the unit circle, always leads to the function $-\sum_1^\infty f(-n)z^{-n}$. The set of the singular points is exactly the closure of the set $\{e^{i\lambda_p}\}$, where $-\lambda_p$ are the exponents of the u.a.p. function which interpolates the sequence a_n "linearly." Isolated singularities are simple poles and correspond to the isolated exponents mod 2π.*

4. The ζ-function

In the two cases we have just considered we started (for $f(t) = 1$) from the simple functions s^{-1} and $(1-z)^{-1}$ respectively. Their simple properties allowed us to give a complete answer to our problem when $f(t)$ is any u.a.p. function. We now turn our attention to the case where these functions are replaced by the ζ-function of Riemann:

$$\zeta(s) = \sum_1^\infty e^{-s\log n}$$

i.e., we consider the continuation of the ordinary Dirichlet series

$$G_1(s) = \sum_1^\infty f(\log n)\, e^{-s\log n},$$

where $f(t) \sim \sum_1^\infty a_p e^{i\lambda_p t}$. When $f(t) = e^{i\lambda t}$ this function becomes $\zeta(s - i\lambda)$; the original function is simply shifted as before. However, the general results cannot be extended to this problem unless some *restrictions are imposed on the function $f(t)$*.

[9] H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, Acta Math. **46** (1925), pp. 101–213.

We begin with the second part, i.e., we first study the growth of $G_1(s)$ when $\sigma \to +0$ for a fixed τ. No difficulties present themselves in this question. We find that, by the methods used previously,

$$\lim_{\sigma \to 1+0} \sigma\, G_1(\sigma + i\tau) = a(\tau),$$

in other words, all points of the set $\mathfrak{M}$, the closure of $\{1 + i\lambda_p\}$, are singular. But other singularities than these can show up, as will be shown by the following example.[10]

Let $a_n = 3^{-n}$, then $\sum_{n+1}^{\infty} a_\nu = \frac{1}{2} a_n$. Let us choose a sequence $\sigma_1, \tau_1, \sigma_2, \tau_2, \cdots$ such that

1) $2 > \sigma_1 > \sigma_2 > \cdots > \sigma_n > \cdots \to 1$,

2) all $\tau_n > 2$,

3) $\frac{1}{2} a_n \zeta(\sigma_n) > K_n + n + 1$, where $K_n = \operatorname{Max} \left| \sum_{1}^{n-1} a_\nu \zeta(s - i\tau_\nu) \right|$ for $|s - 1| < 1$,

4) $a_n |\zeta(\sigma_n - i\tau_n)| > a_n \zeta(\sigma_n) - 1$ (possible because of the almost periodicity of $\zeta(s)$ on any line $\sigma = \sigma_0 > 1$).

Such a sequence exists, since the conditions are of an inductive type: having determined $\sigma_1, \tau_1, \cdots, \sigma_{n-1}, \tau_{n-1}$, we find σ_n by condition 3 and then find τ_n by condition 4. We form the u.a.p. function $\sum_{1}^{\infty} a_n e^{i\tau_n t}$; its exponents τ_n are all > 2. Due to the absolute convergence of the series Σa_n we find that, for any s in $\sigma > 0$,

$$G_1(s) = \sum_{1}^{\infty} a_n \zeta(s - i\tau_n).$$

But

$$G_1(\sigma_n) = \sum_{1}^{n-1} a_\nu \zeta(\sigma_n - i\tau_\nu) + a_n \zeta(\sigma_n - i\tau_n) + \sum_{n+1}^{\infty} a_\nu \zeta(\sigma_n - i\tau_\nu),$$

and therefore

$$|G_1(\sigma_n)| \geqq -K_n + a_n \zeta(\sigma_n) - 1 - \sum_{n+1}^{\infty} a_\nu \zeta(\sigma_n) > n.$$

The point $s = 1$ is singular, although 0 is not a limit point of the exponents of $f(t)$.

We must therefore restrict the function $f(t)$. We shall require that the func-

[10] We are indebted to Dr. Jessen for this example.

tion has *two u.a.p. derivatives.* We have to integrate over the whole straight line $\sigma = 1$. To guarantee absolute convergence of the integrals involved, we use the kernel

$$K_1(s, t) = 1/2\pi \int_{-\infty}^{+\infty} \zeta(s - \varphi(\lambda))\, \varphi^{-2}(\lambda)\, e^{i\lambda t}\, d\lambda, \qquad \sigma > 1,$$

where $\varphi(\lambda) = i\lambda$ for $|\lambda| \geqq 1$ and where $\varphi(\lambda)$ avoids the origin. The inversion gives

$$\zeta(s - \varphi(\lambda)) = \varphi(\lambda)^2 \int_{-\infty}^{+\infty} K_1(s, t)\, e^{-\lambda t}\, dt.$$

The function

$$G_1(s) = \int_{-\infty}^{+\infty} K_1(s, t)\, f''(t)\, dt$$

exists and forms the analytic continuation of $\Sigma f(\log n) e^{-s \log n}$ in the neighborhood of $s = 1$

THEOREM 5. *If the u.a.p. function possesses two u.a.p. derivatives, then the continuation of* $\Sigma f(\log n) e^{-s \log n}$ *leads to a uniform function. Its singular points fill out exactly the closure of the set* $\{1 + i\lambda_p\}$. *Isolated singularities are always simple poles; they correspond to the isolated exponents.*

The conditions of this theorem will be satisfied, for example, for any u.a.p. function whose exponents are absolutely bounded, or more generally in the case where $\sum_{\lambda} \frac{|a_\lambda|}{\lambda^2}$ is convergent for large $|\lambda|$.

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CAUCHY'S PROBLEM FOR SYSTEMS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

BY EDWIN W. TITT[1]

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1. Introduction. É. Cartan[2] has obtained a canonical form for systems of first order partial differential equations which are linear in the first derivatives. He gave existence theorems of the Cauchy type for such systems when in involution and defined their characteristic surfaces. Hadamard[3] has made a study of systems of second order equations of the Cauchy-Kowalewsky type giving existence theorems which show that Cauchy's problem for a characteristic is indeterminate.

It is the purpose of the present paper to obtain a canonical form for systems of second order partial differential equations, linear in the second derivatives. Although the ideas of Cartan are utilized in obtaining this canonical form, new elements enter into the consideration of the second order problem; for example, in the first order case there is only one possibility for a canonical form while in the second order case it is not evident which of two possibilities will give the desired results. Making use of Riquier's theory of orthonomic systems, we are then led to a clear statement of Cauchy's problem for the systems under consideration. That the original system and the system obtained by differentiating once satisfy a certain number of identities is found to be a necessary and sufficient condition for the complete integrability of the equivalent canonical form. The characteristic surfaces are the surfaces over which our results on Cauchy's problem fail to apply. The theory is applied to Einstein's gravitational equations for free space, and the theory of normal systems of differential equations of Thomas and Titt[4] (an extension of Riquier's work) is employed in obtaining existence theorems which show that Cauchy's problem for a characteristic surface is indeterminate. A system of first order equations equivalent to Einstein's equations has been considered by Thomas and Titt employing the methods of Cartan and Thomas.[5] The present treatment does not make use of the theory of differential invariants of generalized spaces. On account of the fact that we are dealing with fewer equations, the consideration of a system of second order equations, as such, simplifies the general theory and reduces the calculations necessary in the consideration of a particular example.

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[1] National Research Fellow.

[2] É. Cartan, *Sur la théorie des systèmes en involution et ses applications à la Relativité*, Bull. Soc. Math. de France, 59, (1931), p. 88.

[3] J. Hadamard, *Leçons sur la propagation des ondes*, (1903), Chap. 7.

[4] Thomas and Titt, *Systems of partial differential equations and their characteristic surfaces*, Ann. of Math., (2), 34, (1933), p. 53.

[5] See *loc. cit.*, Thomas and Titt for further references.

2. **Regular Systems.** Consider a system of L partial differential equations, linear in the second derivatives of w unknowns $v_1, \cdots, v_w$ with respect to n independent variables $x^1, \cdots, x^n$, namely,

$$Z_i \equiv \sum_{h=1}^{w} \sum_{\alpha,\beta=1}^{n} a_i^{h\alpha\beta} \frac{\partial^2 v_h}{\partial x^\alpha \partial x^\beta} + c_i = 0 \qquad (i = 1, \cdots, L). \tag{2.1}$$

The coefficients $a_i^{h\alpha\beta}$ and c_i are functions of the independent variables x^α, the unknowns v_h, and the first derivatives of the unknowns $\partial v_h/\partial x^\alpha$. Let us also assume that the L equations are linearly independent in the second derivatives $\partial^2 v_h/\partial x^\alpha \partial x^\beta$.

Suppose that L_{11} of the equations (2.1) are independent in the derivatives $\partial^2 v_h/\partial x^1 \partial x^1$ in a neighborhood containing the set of values $x^\alpha = 0$. Since the number L_{11} may be dependent on the coordinate system x^α, let us suppose that the coordinates x^α have been so selected that L_{11} has its maximum value. Then replace the system (2.1) by a system containing L_{11} equations independent in the derivatives $\partial^2 v_h/\partial x^1 \partial x^1$ and a set S_1 consisting of $L - L_{11}$ equations free from derivatives $\partial^2 v_h/\partial x^1 \partial x^1$. Suppose that L_{12} equations of the set S_1 are independent in the derivatives $\partial^2 v_h/\partial x^1 \partial x^2$, in fact let us suppose that out of the coordinate systems in which L_{11} has its maximum value we select one in which L_{12} has its maximum value. Repeating this process in the order $L_{11}, L_{12}, L_{22}, L_{13}, \cdots, L_{nn}$, we finally arrive at a system

$$\sum_{h=1}^{w} \sum_{\alpha=1}^{\beta} \sum_{\beta=1}^{n} b_{irs}^{h\alpha\beta} \frac{\partial^2 v_h}{\partial x^\alpha \partial x^\beta} + c_{irs} = 0 \qquad \begin{bmatrix} i = 1, \cdots, L_{rs} \\ r = 1, \cdots, s \\ s = 1, \cdots, n \end{bmatrix} \tag{2.2}$$

equivalent to (2.1), in which $b_{irs}^{h\alpha\beta} = 0$ if $\beta < s$ or if $\beta = s$ and $\alpha < r$. A system of coordinates x^α in which the system (2.1) can be put in the form (2.2) will be said to be *non-singular*; otherwise the coordinate system x^α will be said to be *singular*.

For the sake of clarity let us consider the case $n = 3$. The method extends in an obvious manner for n arbitrary. If $n = 3$, the system (2.2) can be written

$$\begin{aligned}
&a_i^{(1)h} \frac{\partial^2 v_h}{\partial x \partial x} + b_i^{(1)h} \frac{\partial^2 v_h}{\partial x \partial y} + c_i^{(1)h} \frac{\partial^2 v_h}{\partial y \partial y} + d_i^{(1)h} \frac{\partial^2 v_h}{\partial x \partial z} + e_i^{(1)h} \frac{\partial^2 v_h}{\partial y \partial z} \\
&\qquad + f_i^{(1)h} \frac{\partial^2 v_h}{\partial z \partial z} + \star = 0 \\
&b_j^{(2)h} \frac{\partial^2 v_h}{\partial x \partial y} + c_j^{(2)h} \frac{\partial^2 v_h}{\partial y \partial y} + d_j^{(2)h} \frac{\partial^2 v_h}{\partial x \partial z} + e_j^{(2)h} \frac{\partial^2 v_h}{\partial y \partial z} + f_j^{(2)h} \frac{\partial^2 v_h}{\partial z \partial z} + \star = 0 \\
&c_k^{(3)h} \frac{\partial^2 v_h}{\partial y \partial y} + d_k^{(3)h} \frac{\partial^2 v_h}{\partial x \partial z} + e_k^{(3)h} \frac{\partial^2 v_h}{\partial y \partial z} + f_k^{(3)h} \frac{\partial^2 v_h}{\partial z \partial z} + \star = 0 \\
&d_l^{(4)h} \frac{\partial^2 v_h}{\partial x \partial z} + e_l^{(4)h} \frac{\partial^2 v_h}{\partial y \partial z} + f_l^{(4)h} \frac{\partial^2 v_h}{\partial z \partial z} + \star = 0
\end{aligned} \tag{2.3}$$

$$e_m^{(5)\lambda}\frac{\partial^2 v_\lambda}{\partial y\partial z}+f_m^{(5)\lambda}\frac{\partial^2 v_\lambda}{\partial z\partial z}+\star=0$$

$$f_n^{(6)\lambda}\frac{\partial^2 v_\lambda}{\partial z\partial z}+\star=0$$

$$\begin{pmatrix} i=1,\cdots,L_{11};\, j=1,\cdots,L_{12};\, k=1,\cdots,L_{22}\\ l=1,\cdots,L_{13};\, m=1,\cdots,L_{23};\, n=1,\cdots,L_{33}\end{pmatrix},$$

where the $\star$ denotes terms depending on derivatives of order lower than the second. Under the transformation

$$\bar{x}=x+\gamma y,\qquad \bar{y}=y+\delta x+\epsilon z,\qquad \bar{z}=z, \tag{2.4}$$

where γ, δ, ϵ are non-negative real constants, we have

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial \bar{x}}+\delta\frac{\partial v}{\partial \bar{y}},\quad \frac{\partial v}{\partial y}=\frac{\partial v}{\partial \bar{y}}+\gamma\frac{\partial v}{\partial \bar{x}},\quad \frac{\partial v}{\partial z}=\frac{\partial v}{\partial \bar{z}}+\epsilon\frac{\partial v}{\partial \bar{y}}$$

and

$$\begin{aligned}
\frac{\partial^2 v}{\partial x\partial x}&=\frac{\partial^2 v}{\partial \bar{x}\partial \bar{x}}+2\delta\frac{\partial^2 v}{\partial \bar{x}\partial \bar{y}}+\delta^2\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}};\\
\frac{\partial^2 v}{\partial x\partial y}&=(1+\gamma\delta)\frac{\partial^2 v}{\partial \bar{x}\partial \bar{y}}+\gamma\frac{\partial^2 v}{\partial \bar{x}\partial \bar{x}}+\delta\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}};\\
\frac{\partial^2 v}{\partial y\partial y}&=\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}}+2\gamma\frac{\partial^2 v}{\partial \bar{x}\partial \bar{y}}+\gamma^2\frac{\partial^2 v}{\partial \bar{x}\partial \bar{x}};\\
\frac{\partial^2 v}{\partial x\partial z}&=\frac{\partial^2 v}{\partial \bar{x}\partial \bar{z}}+\epsilon\frac{\partial^2 v}{\partial \bar{x}\partial \bar{y}}+\delta\frac{\partial^2 v}{\partial \bar{y}\partial \bar{z}}+\epsilon\delta\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}};\\
\frac{\partial^2 v}{\partial y\partial z}&=\frac{\partial^2 v}{\partial \bar{y}\partial \bar{z}}+\gamma\epsilon\frac{\partial^2 v}{\partial \bar{x}\partial \bar{y}}+\epsilon\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}}+\gamma\frac{\partial^2 v}{\partial \bar{x}\partial \bar{z}};\\
\frac{\partial^2 v}{\partial z\partial z}&=\frac{\partial^2 v}{\partial \bar{z}\partial \bar{z}}+2\epsilon\frac{\partial^2 v}{\partial \bar{y}\partial \bar{z}}+\epsilon^2\frac{\partial^2 v}{\partial \bar{y}\partial \bar{y}}.
\end{aligned} \tag{2.5}$$

First put $\delta=\epsilon=0$ in (2.4). We shall adopt the notation

$$\left\| a_i^{(1)\lambda}+\gamma b_i^{(1)\lambda}+\gamma^2 c_i^{(1)\lambda}\right\| \qquad \begin{pmatrix}\lambda=1,\cdots,w\\ i=1,\cdots,L_{11}\end{pmatrix} \tag{2.6}$$

for the matrix of the coefficients of the derivatives $\partial^2 v_\lambda/\partial\bar{x}\partial\bar{x}$ in the first L_{11} transformed equations (2.3), and a similar notation for matrices and determinants will be employed throughout the remainder of the paper. If the constant γ is chosen small enough, the matrix (2.6) will be of maximum rank, i.e., of rank L_{11}. Hence we must have L_{12} linear relations of the form

$$\gamma b_j^{(2)h} + \gamma^2 b_j^{(2)h} = \sum_{i=1}^{L_{11}} \lambda_j^i \, (a_i^{(1)h} + \gamma b_i^{(1)h} + \gamma^2 c_i^{(1)h}); \tag{2.7}$$

otherwise the coordinate system x^α would be singular, contrary to hypothesis. Now divide equations (2.7) through by γ and, allowing γ to approach zero, we have

$$b_j^{(2)h} = \sum_{i=1}^{L_{11}} \lambda_j^{(1)i} \, a_i^{(1)h} \qquad \begin{pmatrix} h = 1, \cdots, w \\ j = 1, \cdots, L_{12} \end{pmatrix}, \tag{2.8}$$

where

$$\lambda_j^{(1)i} = \lim_{\gamma \to 0} \left(\frac{\lambda_j^i}{\gamma} \right).$$

In a similar manner we have, under the hypothesis $\delta = \epsilon = 0$, the linear relations

$$c_k^{(3)h} = \sum_{j=1}^{L_{12}} \lambda_k^{(2)j} \, b_j^{(2)h} \qquad \begin{pmatrix} h = 1, \cdots, w \\ k = 1, \cdots, L_{22} \end{pmatrix} \tag{2.9}$$

$$e_m^{(5)h} = \sum_{l=1}^{L_{12}} \lambda_m^{(4)l} \, d_l^{(4)h} \qquad \begin{pmatrix} h = 1, \cdots, w \\ m = 1, \cdots, L_{23} \end{pmatrix}. \tag{2.10}$$

Under the hypothesis $\gamma = \delta = 0$, we have

$$f_n^{(6)h} = \sum_{m=1}^{L_{23}} \lambda_n^{(5)m} \, e_m^{(5)h} \qquad (n = 1, \cdots, L_{33}), \tag{2.11}$$

$$e_l^{(4)h} = \sum_{k=1}^{L_{22}} \lambda_l^{*k} \, c_k^{(3)h} \qquad (l = 1, \cdots, L_{13}). \tag{2.12}$$

Then assuming $\gamma = 0$, we have

$$\epsilon\delta \, d_l^{(4)h} + \epsilon \, e_l^{(4)h} + \epsilon^2 f_l^{(4)h} = \sum_{k=1}^{L_{22}} \lambda_l'^k \, (c_k^{(3)h} + \epsilon\delta \, d_k^{(3)h} + \epsilon \, e_k^{(3)h} + \epsilon^2 f_k^{(3)h})$$

$$(l = 1, \cdots, L_{13}),$$

if the constants ϵ and δ are chosen small enough; otherwise the coordinate system x^α would be singular. Dividing by ϵ and allowing ϵ to approach zero, we have

$$\delta \, d_l^{(4)h} + e_l^{(4)h} = \sum_{k=1}^{L_{22}} \lambda_l^{**k} \, c_k^{(3)h} \qquad (l = 1, \cdots, L_{13}). \tag{2.13}$$

Substitute from (2.12) into (2.13) and we obtain the desired relation

$$d_l^{(4)h} = \sum_{k=1}^{L_{22}} \lambda_l^{(3)k} \, c_k^{(3)h} \qquad (l = 1, \cdots, L_{13}). \tag{2.14}$$

Taking account of (2.8) and the fact that the matrices

$$\|a_i^{(1)h}\|, \|b_j^{(2)h}\| \qquad \begin{pmatrix} h = 1, \cdots, w \\ i = 1, \cdots, L_{11} \\ j = 1, \cdots, L_{12} \end{pmatrix}$$

are of maximum rank, it follows that

$$L_{11} \geqq L_{12}.$$

Now let us suppose that the determinant

$$|a_i^{(1)h}| \qquad (h, i = 1, \cdots, L_{11}) \tag{2.15}$$

is different from zero; this can be accomplished by a relettering of the unknowns v_h. On account of (2.8), the rank R of the matrix

$$\|\lambda_j^{(1)i}\| \qquad \begin{pmatrix} i = 1, \cdots, L_{11} \\ j = 1, \cdots, L_{12} \end{pmatrix}$$

is greater than or equal to L_{12}.[6] But since the range of the index j is $1, \cdots, L_{12}$, the rank R must be equal to L_{12}. Taking account of (2.8) again we have then that the rank of the matrix

$$\|b_j^{(2)h}\| \qquad \begin{pmatrix} h = 1, \cdots, L_{11} \\ j = 1, \cdots, L_{12} \end{pmatrix}$$

is equal to L_{12};[7] in fact we can assume that the determinant

$$|b_j^{(2)h}| \qquad (h, j = 1, \cdots, L_{12}) \tag{2.16}$$

is different from zero. Upon repeating this argument for equations (2.9), (2.14), (2.10), and (2.11), it follows that

$$L_{11} \geqq L_{12} \geqq L_{22} \geqq L_{13} \geqq L_{23} \geqq L_{33},$$

and also that the determinants

$$\begin{aligned} &|c_k^{(3)h}| \qquad (h, k = 1, \cdots, L_{22}), \qquad |d_l^{(4)h}| \qquad (h, l = 1, \cdots, L_{13}), \\ &|e_m^{(5)h}| \qquad (h, m = 1, \cdots, L_{23}), \qquad |f_n^{(6)h}| \qquad (h, n = 1, \cdots, L_{33}), \end{aligned} \tag{2.17}$$

can be made different from zero.

Because of the non-vanishing of (2.15), (2.16), (2.17), and the corresponding determinants for n arbitrary, the system (2.1) can be replaced by a system in the following *regular* form:[8]

$$\frac{\partial^2 v_i}{\partial x^r \partial x^s} = \sum_{h=1}^{w} \sum_{\alpha=1}^{\beta} \sum_{\beta=1}^{n} g_{i\,r\,s}^{h\,\alpha\,\beta} \frac{\partial^2 v_h}{\partial x^\alpha \partial x^\beta} + \star \qquad \begin{pmatrix} i = 1, \cdots, L_{rs} \\ r = 1, \cdots, s \\ s = 1, \cdots, n \end{pmatrix}, \tag{2.18}$$

---

[6] See Bocher, *Introduction to Higher Algebra*, p. 78.

[7] *Loc. cit.*, Bocher, p. 79.

[8] The term *regular* is used by analogy with the regular, linear, first order systems of Méray and Riquier, *Sur la convergence des développements des intégrales ordinaires*, Ann. Ec. Norm. Sup., (3), 7, (1890), p. 44.

in which no left member appears on the right of any equation and $g_{i\ r\ s}^{h\ \alpha\ \beta} = 0$ if $s > \beta$, or if $s = \beta$ and $r > \alpha$. The numbers L_{rs} satisfy the inequalities

$$L_{11} \geqq L_{12} \geqq L_{22} \geqq L_{13} \geqq \cdots \geqq L_{nn}.$$

THEOREM I. *In any non-singular coordinate system, the system* (2 1) *can be put in the regular form* (2.18).

The proof that any system of the form (2.1) can be put in regular form can be extended to systems in which the unknowns are the components of a tensor or an affine connection on account of the linearity of the transformation (2.4).[9]

3. **Cauchy's Problem.** If the coefficients $a_i^{h\ \alpha\ \beta}$ and c_i in (2.1) are analytic functions of their arguments in the neighborhood of the set of values

$$x^\alpha = 0, \qquad v_h = (v_h)_0, \qquad \frac{\partial v_h}{\partial x^\alpha} = \left(\frac{\partial v_h}{\partial x^\alpha}\right)_0, \tag{3.1}$$

it follows that the equivalent regular system (2.18) is *orthonomic*.[10] In fact, let us give the variables the following *marks*:

(3.2)

Mark	x^1	x^2	$\cdots$	x^{n-1}	x^n	v_1	$\cdots$	v_w
1	1	1	$\cdots$	1	1	0	$\cdots$	0
2	$2n$	$2n-2$	$\cdots$	4	2	0	$\cdots$	0
3	0	0	$\cdots$	0	0			

It will be found convenient to adopt a new notation for the unknowns v_h and assign their third marks as follows:

(3.3)

Group	Notation	Third Mark
$v_{L_{11}+1}, \cdots, v_w$	v_{k00}	0
$v_{L_{12}+1}, \cdots, v_{L_{11}}$	v_{k11}	1
.	.	.
.	.	.
.	.	.
$v_{L_{nn}+1}, \cdots, v_{L_{(n-1)n}}$	$v_{k(n-1)n}$	$K(n,2)-1$
$v_1, \cdots, v_{L_{nn}}$	v_{knn}	$K(n,2)$[11]

[9] Cf. *loc. cit.*, Thomas and Titt, §3.

[10] Riquier, *Les systèmes d'équations aux dérivées partielles*, (1910), p. 201. If in the definition of a *normal* system of Thomas and Titt, *loc. cit.*, p. 53, we impose the condition that the system S is vacuous, then we have the definition of an orthonomic system; hence in what follows we shall refer to this latter treatment for the theory of orthonomic systems. In order to understand the present paper one need read only the self contained theory of normal systems in §§14–18. In what follows we shall use the terms *monomial, class, multiplier, non-multiplier, set, complementary set, complete, strongly complete, precede, initial determinations, conditions of integrability, completely integrable, parametric derivative,* and *principle derivative* without further reference. The term *cote* used by Riquier will be translated *mark* following Ritt, *Algebraic differential equations*, Amer. Math. Soc., (1932), p. 141.

[11] $K(n, 2)$ is used to denote the number of combinations of n things taken two at a time, repetitions allowed.

The index k on the unknown v_{kab} takes on the range $1, \cdots, M_{ab}$, where M_{ab} is the number of unknowns v_h in the corresponding group in (3.3). With this new notation, equations (2.18) can be written

$$X_{kabrs} \equiv \frac{\partial^2 v_{kab}}{\partial x^r \, \partial x^s} - g^{lcduv}_{kabrs} \frac{\partial^2 v_{lcd}}{\partial x^u \, \partial x^v} + \star = 0 \tag{3.4}$$

$$\begin{pmatrix} b, v = 1, \cdots, n; a, s = 1, \cdots, b; r = 1, \cdots, s; \text{ if } s = b, r \leqq a; k = 1, \cdots, M_{ab}. \\ u = 1, \cdots, v; v \geqq s; \text{ if } v = s, u \geqq r; v \geqq d; \text{ if } v = d, u > c. \end{pmatrix}$$

From (3.2) it can be seen that the first marks of the derivatives imply that a first derivative *precedes* a second derivative; the second marks imply that all the derivatives on the right of (3.4) precede their corresponding left members unless $v = s$, $u = r$; the third marks complete the ordering for an orthonomic system, i.e., every derivative on the right of an equation (3.4) precedes the corresponding left member. By inspection of the following table of *multipliers* and *non-multipliers*, it is evident that the *set* containing $K(n, 2)$ *monomials* is *complete*, and furthermore that any subset obtained by taking the first α monomials $[\alpha = 1, \cdots, K(n, 2)]$ is also complete.

(3.5)

Monomial	Multipliers	Non-multipliers
x_1^2	$x_1, \cdots, x_n$	
$x_1 x_2$	$x_2, \cdots, x_n$	x_1
x_2^2	$x_2, \cdots, x_n$	x_1
$x_1 x_3$	$x_3, \cdots, x_n$	x_1, x_2
$x_2 x_3$	$x_3, \cdots, x_n$	x_1, x_2
x_3^2	$x_3, \cdots, x_n$	x_1, x_2
.	.	.
.	.	.
.	.	.
$x_1 x_n$	x_n	$x_1, \cdots, x_{n-1}$
.	.	.
.	.	.
.	.	.
x_n^2	x_n	$x_1, \cdots, x_{n-1}$

This completes the proof that the regular system (3.4) is orthonomic and, assuming (3.4) to be *completely integrable*, let us seek the form of the arbitrary data. To do so we must consider the form of the *complementary sets* to the sets (3.5) as illustrated in the following table.

(3.6)

Set M	Complementary Sets	Multipliers
x_1^2	N_1 x_1	$x_2, \cdots, x_n$
	N_1 1	$x_2, \cdots, x_n$
x_1^2	$N_1, \cdots, N_{b-2}$ are vacuous	
$x_a x_b$ $(a \neq b)$	N_{b-1} x_{a+1}	$x_b, \cdots, x_n$
	N_{b-1} x_{b-1}	
	N_{b-1} 1	$x_b, \cdots, x_n$
	N_b x_1	$x_{b+1}, \cdots, x_n$
	N_b x_a	$x_{b+1}, \cdots, x_n$
x_1^2	$N_1, \cdots, N_{b-1}$ are vacuous	
x_b^2	N_b x_1	$x_{b+1}, \cdots, x_n$
	N_b x_b	
	N_b 1	$x_{b+1}, \cdots, x_n$

Thus the *initial determinations* for the unknowns $v_{k\,a\,b}$ can be represented as follows:

$$v_{k00} \cdots \text{Completely Arbitrary}$$

$$v_{kbb} \cdots \begin{cases} v_{kbb} & F(x^{b+1}, \cdots, x^n) \\ \dfrac{\partial v_{kbb}}{\partial x^i} & \text{over } x^1 = 0, \cdots, x^b = 0 \\ & (i = 1, \cdots, b) \end{cases}$$

$$
(3.7)\qquad
\begin{aligned}
&v_{kab} \quad \{\ F(x^b, , \cdots x^n)\\
&\frac{\partial v_{kab}}{\partial x^j} \quad \{\ \text{over } x^1 = 0, \cdots, x^{b-1} = 0\\
&\qquad\qquad\quad (j = a+1, \cdots, b-1)\\
&v_{kab}\ (a \neq b)\\
&\frac{\partial v_{kab}}{\partial x^m} \quad \{\ F(x^{b+1}, \cdots, x^n)\\
&\qquad\qquad\quad \text{over } x^1 = 0, \cdots, x^b = 0\\
&\qquad\qquad\quad (m = 1, \cdots, a).
\end{aligned}
$$

EXISTENCE THEOREM: *If the arbitrary functions in* (3.7) *be assigned analytic in the neighborhood of* $x^\alpha = 0$, *then there exists one, and only one, set of functions* $v_{kab}(x)$ $(a, b \neq 0)$, *analytic in the neighborhood of* $x^\alpha = 0$, *which together with their first derivatives reduce to the assigned functions* (3.7) *over the indicated surfaces, and which together with the* v_{k00} (x) *constitute a solution of a completely integrable system* (3.4).

In the above theorem we are looking upon the completely arbitrary functions v_{k00} (x) as known functions and the system (2.1) as a system of differential equations for the determination of the unknowns v_{kab} $(a, b \neq 0)$. By Cauchy's problem for the surface $x^1 = 0$, we shall mean the problem of determining the unknowns v_{kab} $(a, b \neq 0)$ from the knowledge of the values of these unknowns and their first derivatives $\partial v_{kab}/\partial x^1$ over the surface $x^1 = 0$. From the above existence theorem it follows that Cauchy's problem for the surface $x^1 = 0$ cannot have more than one analytic solution. Conversely, if the unknowns $v_{kab}(a, b \neq 0)$ and their first derivatives are assigned over the surface $x^1 = 0$ as indicated by (3.7), then Cauchy's problem has a unique analytic solution.

4. **A Condition for Complete Integrability.** It is now important to find the relation between the identities which the system (2.1) satisfies and the complete integrability of the equivalent regular system (3.4). It must be possible to write equations (2.1) in the form

$$
(4.1)\qquad Z_i \equiv c_i^{kabrs} X_{kabrs} = 0 \qquad (i = 1, \cdots, L),
$$

where the quantities X_{kabrs} are given by (3.4), and the matrix

$$
(4.2)\qquad \| c_i^{kabrs}(x, v, \partial v/\partial x) \|
$$

is non-singular.

There is one *integrability condition* of (3.4) corresponding to each combination of one of the monomials of table (3.5) with one of its non-multipliers. The condition for completeness of a set of monomials is that each product of a monomial of the set by one of its non-multipliers be equal to the product of another monomial of the set by multipliers of the latter. This pairing of monomials can be represented in terms of derivatives of the left members of the system (3.4) as follows:

$$\frac{\partial X_{kabrs}}{\partial x^t},\ \frac{\partial X_{kabrt}}{\partial x^s} \qquad \begin{pmatrix} t < r \leqq s \leqq b \\ r \leqq t < s \leqq b \\ \text{if } s = b,\ r \leqq a \end{pmatrix}, \tag{4.3}$$

where it is understood that the second term of (4.3) is to mean $\partial X_{kabtr}/\partial x^s$ in case $t < r$.

From (3.4) we have

$$\frac{\partial X_{kabrs}}{\partial x^\beta} \equiv \frac{\partial^2 v_{kab}}{\partial x^r \partial x^s \partial x^\beta} - g^{lcduv}_{kabrs}\ \frac{\partial^2 v_{lcd}}{\partial x^u \partial x^v \partial x^\beta} + \star = 0 \tag{4.4}$$

$$(v \geqq s;\ \text{if } v = s,\ u \geqq r;\ v \geqq d;\ \text{if } v = d,\ u > c).$$

The property that every derivative on the right of an equation (3.4) precedes the corresponding left member with the assignment of marks given by (3.2) and (3.3) is preserved under differentiation. Consequently the system (4.4) can be replaced by a system in which the summation in (4.4) is replaced by a sum containing only *parametric derivatives* for (3.4). Thus we have[12]

$$Y_{kabrs\beta} \equiv \frac{\partial^2 v_{kab}}{\partial x^r \partial x^s \partial x^\beta} - h^{lcduvw}_{kabrs\beta}\ \frac{\partial^2 v_{lcd}}{\partial x^u \partial x^v \partial x^w} + \star = 0 \tag{4.5}$$

$$(u \leqq v \leqq w;\ v \geqq d;\ \text{if } v = d,\ u > c).$$

Furthermore there exist relations of the form

$$\frac{\partial X_{kabrs}}{\partial x^\beta} \equiv d^{KABRST}_{kabrs\beta}\, Y_{KABRST}, \tag{4.6}$$

where K, A, B, R, S, T have the same ranges respectively as k, a, b, r, s, β; the matrix

$$\| d^{KABRST}_{kab,s\beta}(x, v, \partial v/\partial x) \| \tag{4.7}$$

is non-singular.

By (4.3) and (4.5), there are then

$$A = \sum_{r=1}^{s} \sum_{s=2}^{n} (s-1)\, L_{rs}$$

conditions of the type

$$Y_{kabrst} - Y_{kabrts} = 0 \qquad \begin{pmatrix} t < r \leqq s \leqq b \\ r \leqq t < s \leqq b \\ \text{if } s = b,\ r \leqq a \end{pmatrix} \tag{4.8}$$

which must be satisfied identically if the system (3.4) is completely integrable; i.e., the equations (4.8) must be satisfied identically when the equations (3.4)

[12] *Loc. cit.*, Thomas and Titt, p. 54.

themselves have been used to eliminate all *principle derivatives* of (3.4) from the $\star$ terms in (4.5). When account is taken of the fact that the matrices (4.2) and (4.7) are non-singular, it follows that there are A linear combinations of the derivatives of the left members of the system (2.1), namely

$$\sum_{i=1}^{L}\sum_{\alpha=1}^{n} \Lambda_\sigma^{i\alpha}\frac{\partial Z_i}{\partial x^\alpha} = 0 \qquad (\sigma = 1, \cdots, A), \tag{4.9}$$

which vanish in consequence of (3.4) or (2.1); the matrix

$$\|\Lambda_\sigma^{i\alpha}(x, v, \partial v/\partial x)\| \tag{4.10}$$

is of maximum rank, i.e., of rank A.

Conversely, suppose that we know A linearly independent identities of the type (4.9) which are satisfied in consequence of (2.1). Our problem will be to show that the regular system (3.4) equivalent to (2.1) is completely integrable. Making use of (4.1) and (4.6), the identities (4.9) can be written

$$\sum_{i=1}^{L}\sum_{\alpha=1}^{n} \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrs\beta}\, Y_{kabrs\beta} + \star = 0 \qquad (\sigma = 1, \cdots, A), \tag{4.11}$$

where the matrix

$$\|\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrs\beta}(x, v, \partial v/\partial x)\| \tag{4.12}$$

is of rank A. Since the equations (4.11) are satisfied in consequence of (2.1) or (3.4), the coefficients of the third derivatives $\partial^3 v_{ka}/\partial x^\alpha \partial x^\beta \partial x^\gamma$ in (4.11) must be identically zero. Hence

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabsss} \equiv 0 \qquad (s \leqq b;\ \text{if } s = b,\ s = a). \tag{4.13}$$

By a similar argument we have

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabsst} + \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabtss} \equiv 0 \qquad \begin{pmatrix} t < s \leqq b \\ \text{if } s = b,\ s = a \end{pmatrix}, \tag{4.14}$$

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabtst} + \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabtts} \equiv 0 \qquad \begin{pmatrix} t < s \leqq b \\ \text{if } s = b,\ t \leqq a \end{pmatrix}, \tag{4.15}$$

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrst} + \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabtsr} + \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrts} \equiv 0 \qquad (r < t < s \leqq b;\ \text{if } s = b,\ t \leqq a), \tag{4.16}$$

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrst} + \Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrts} \equiv 0 \qquad (r < t < s = b;\ r \leqq a < t). \tag{4.17}$$

By equations (4.13), (4.14), (4.15), (4.16), and (4.17), equations (4.11) can be written

$$\Lambda_\sigma^{i\alpha}\, e_{i\alpha}^{kabrst}(Y_{kabrst} - Y_{kabrts}) + \Sigma\, Y_{lcduvw} + \star = 0 \tag{4.18}$$

$(t < r \leqq s \leqq b;\ \text{or } r \leqq t < s \leqq b;\ \text{if } s = b,\ r \leqq a;\ w \geqq d;\ \text{if } w = d,\ v > c).$

Now suppose that equations (4.18) are not linearly independent in the left members of (4.8); then replace the system (4.18) by an equivalent system containing one equation of the form

$$E^{kabrs\beta}\, Y_{kabrs\beta} + \star = 0 \quad (\beta \geqq b; \text{ if } \beta = b, s > a). \tag{4.19}$$

Since the equation (4.19) must be satisfied in consequence of (3.4), it follows from the form of (4.5) that the coefficients $E^{kabrs\beta}$ must be identically zero, i.e., equations (4.9) are not independent, contrary to hypothesis. Consequently our assumption that the equations (4.18) are not linearly independent in the left members of (4.8) leads to a contradiction. In other words, the existence of A independent identities (4.9) implies that (4.8) are satisfied in consequence of the equations

$$Y_{kabrs\beta} = 0 \qquad (\beta \geqq b; \text{ if } \beta = b, s > a)$$

and the system (3.4), i.e., the system (3.4) is completely integrable.

THEOREM II. *A necessary and sufficient condition for the complete integrability of a regular system* (3.4) *is that there exist A independent linear combinations of the first derivatives of the left members of* (2.1) *which vanish in consequence of* (2.1), *where A is the number of integrability conditions of* (3.4).

Besides being of great importance in the application of the theory of regular systems, Theorem II is important theoretically in that it shows that all regular systems obtained from the same system (2.1) are equivalent in the sense that if one is completely integrable the same will be true of the others.

From the method used in proving the sufficiency of the condition of Theorem II, it follows that there cannot exist more than A independent identities (4.9). This fact will be made use of in §6.

5. **Characteristic Surfaces.** *A surface C_{n-1} having as its equation $x^1 = 0$ will be called an $(n-1)$ dimensional characteristic surface for the system* (2.1) *if it is impossible to find a coordinate system $\bar{x}^\alpha$ defined by a transformation of the form*

$$x^1 = \bar{x}^1, \qquad x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n) \qquad (i = 2, \cdots, n), \tag{5.1}$$

in which the system (2.1) *can be solved for L_{11} derivatives $\partial^2 v_h/\partial\bar{x}^1\partial\bar{x}^1$ at some point P on C_{n-1}.* If the coefficients $a_i^{h\alpha\beta}$ in (2.1) depend on the unknowns v_h or their derivatives, then the characteristic surfaces will depend on a solution $v_h(x)$. It follows from the definition that the results obtained in §3 with regard to Cauchy's problem will not be valid if the arbitrary data is assigned over a characteristic surface C_{n-1}.

Suppose that $\Phi(x^1, \cdots, x^n) = 0$ is the equation of some characteristic surface C_{n-1} of the system (2.1). Under the transformation of coordinates

$$\bar{x}^1 = \Phi(x), \qquad \bar{x}^i = x^i \qquad (i = 2, \cdots, n)^{13} \tag{5.2}$$

[13] In order that (5.2) shall define a transformation of coordinates we must assume that the variables x^α have been relettered so that $\partial\Phi/\partial x^1 \neq 0$ at $x^\alpha = 0$.

the equation of the characteristic surface C_{n-1} assumes the form $\bar{x}^1 = 0$ and the system (2.1) becomes

$$\sum_{h=1}^{w} \sum_{\gamma,\delta=1}^{n} \bar{a}_i^{h\gamma\delta} \frac{\partial^2 v_h}{\partial \bar{x}^\gamma \partial \bar{x}^\delta} + \star = 0 \qquad (i = 1, \cdots, L), \tag{5.3}$$

where

$$\bar{a}_i^{h\gamma\delta} = a_i^{h\alpha\beta} \frac{\partial \bar{x}^\gamma}{\partial x^\alpha} \frac{\partial \bar{x}^\delta}{\partial x^\beta}. \tag{5.4}$$

The fact that $\bar{x}^1 = 0$ is a characteristic surface for the system (5.3) implies that the r determinants that can be formed by taking L_{11} rows and L_{11} columns from the matrix

$$\|\bar{a}_i^{h11}\| \qquad \begin{pmatrix} h = 1, \cdots, w \\ i = 1, \cdots, L \end{pmatrix},$$

namely $\bar{W}_1, \cdots, \bar{W}_r$, vanish over $\bar{x}^1 = 0$. For the transformation (5.2) the equations (5.4) assume the form

$$\bar{a}_i^{h11} = a_i^{h\alpha\beta} \Phi_\alpha \Phi_\beta,$$

where Φ_α is used to denote $\partial\Phi/\partial x^\alpha$. Since $\bar{a}_i^{h11}$ is the component in the $\bar{x}^\alpha$ coordinate system of the scalar $a_i^{h\alpha\beta}\Phi_\alpha\Phi_\beta$, the fact that $\bar{W}_1 = \cdots = \bar{W}_r = 0$ over $\bar{x}^1 = 0$ implies that the corresponding determinants formed from the matrix

$$\|a_i^{h\alpha\beta} \Phi_\alpha \Phi_\beta\| \qquad \begin{pmatrix} h = 1, \cdots, w \\ i = 1, \cdots, L \end{pmatrix}, \tag{5.5}$$

namely $W_1, \cdots, W_r$, vanish over $\Phi = 0$.

Conversely suppose that

$$W_1 = 0, \cdots, W_r = 0 \text{ over } \Phi = 0. \tag{5.6}$$

Then when we make the transformation (5.2), the quantities $\bar{W}_1, \cdots, \bar{W}_r$ will vanish over $\bar{x}^1 = 0$; hence it will be impossible to solve equations (5.3) for L_{11} derivatives $\partial^2 v_h/\partial\bar{x}^1\partial\bar{x}^1$. Furthermore, under the transformations of the form (5.1), namely

$$\bar{x}^1 = \bar{x}^1, \qquad \bar{x}^i = f^i(\bar{x}^1, \cdots, \bar{x}^n) \qquad (i = 2, \cdots, n),$$

we have

$$\tilde{a}_i^{h11} = \bar{a}_i^{h11}.$$

Thus we can state

THEOREM III. *A necessary and sufficient condition that a surface $\Phi = 0$ be an $(n-1)$ dimensional characteristic surface C_{n-1} for the system (2.1) is that all the determinants $W_1, \cdots, W_r$ of order L_{11} formed from the matrix (5.5) vanish over $\Phi = 0$.*

This implies that the equations (5.6) constitute a system of first order partial differential equations for the determination of the unknown Φ. In case the system (2.1) is of the Cauchy-Kowalewsky type, there is only one first order equation (5.6) for the determination of the characteristic surfaces. It is to be noticed that in case the unknowns v_h in the system (2.1) are the components of a tensor the above theorem does not apply.[14]

A surface $C_{n-\alpha}$ having the equation $x^1 = 0, \cdots, x^\alpha = 0$ $(\alpha = 1, \cdots, n-1)$ will be called an $(n-\alpha)$ dimensional characteristic surface of type $\beta(\leqq \alpha)$ for the system (2.1) *if it is impossible to find a coordinate system $\bar{x}^\alpha$ defined by the transformation*

$$x^i = \bar{x}^i, \qquad x^j = f^j(\bar{x}^1, \cdots, \bar{x}^n) \qquad \begin{pmatrix} i = 1, \cdots, \alpha \\ j = \alpha+1, \cdots, n \end{pmatrix} \tag{5.7}$$

in which the system (2.1) *can be solved for*

$$L_{11} + L_{12} + L_{22} + L_{13} + \cdots + L_{\beta\alpha}$$

derivatives

$$\frac{\partial^2 v_h}{\partial \bar{x}^1\, \partial \bar{x}^1}, \frac{\partial^2 v_h}{\partial \bar{x}^1\, \partial \bar{x}^2}, \cdots, \frac{\partial^2 v_h}{\partial \bar{x}^\beta\, \partial \bar{x}^\alpha}$$

at some point on $C_{n-\alpha}$.

A theorem corresponding to Theorem III can be obtained for $(n-\alpha)$ dimensional characteristic surfaces.

6. **Application to Einstein's Equations.** Let us consider Einstein's gravitational equations for free space, namely

$$R_{\alpha\beta} \equiv \tfrac{1}{2} g^{\gamma\delta}\left[\frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta}\right] + \star = 0, \tag{6.1}$$

as an example illustrating the above theory. The coefficients of the second derivatives in the system (6.1) are exhibited in Table I, in which each row corresponds to an equation; any element in Table I is the coefficient of the derivative at the top of the column in which it appears. The notation

$$\alpha = g^{11}; \beta = g^{22}; \gamma = g^{33}; \delta = g^{44}$$

$$a = g^{12}; b = g^{13}; c = g^{14}; d = g^{23}; e = g^{24}; f = g^{34}$$

has been adopted. Evidently $L_{11} = 6$. Since the columns 12, 13, and 14 are porportional to the columns 5, 6, 7, respectively, we need consider only columns 16, 17, 18, 19, 20 in determining L_{12}. Since the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 16 & 17 & 18 & 28 \end{vmatrix} \tag{6.2}$$

[14] Cf. *loc. cit.*, Thomas and Titt, p. 39.

	$\frac{\partial^2}{\partial x^1 \partial x^1}$										$\frac{\partial^2}{\partial x^1 \partial x^2}$										$\frac{\partial^2}{\partial x^2 \partial x^2}$										
	g_{11}	g_{12}	g_{13}	g_{14}	g_{22}	g_{23}	g_{24}	g_{33}	g_{34}	g_{44}	g_{11}	g_{12}	g_{13}	g_{14}	g_{22}	g_{23}	g_{24}	g_{33}	g_{34}	g_{44}	g_{11}	g_{12}	g_{13}	g_{14}	g_{22}	g_{23}	g_{24}	g_{33}	g_{34}	g_{44}	
R_{11}					β	$2d$	$2e$	γ	$2f$	δ		-2β	$-2d$	$-2e$							β										1
R_{12}					$-a$	$-b$	$-c$					$2a$	b	c		d	e	γ	$2f$	δ	$-a$		$-d$	$-e$							2
R_{13}						$-a$		$-b$	$-c$				a			$-\beta$		$-d$	$-e$				β								3
R_{14}							$-a$		$-b$	$-c$				a			$-\beta$		$-d$	$-e$											4
R_{22}					α							-2α				$-2b$	$-2c$				α		$2b$	$2c$				γ	$2f$	δ	5
R_{23}						α							$-\alpha$			a		$-b$	$-c$				$-a$					$-d$	$-e$		6
R_{24}							α							$-\alpha$			a		$-b$	$-c$				$-a$					$-d$	$-e$	7
R_{33}								α										$2a$										β			8
R_{34}									α										$2a$										β		9
R_{44}										α										$2a$										β	10
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	

TABLE I

has the value

$$2\alpha^2 \begin{vmatrix} \alpha & a \\ a & \beta \end{vmatrix} \begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix}^2, \tag{6.3}$$

disregarding algebraic sign,[15] there must be a ninth order minor, σ, in the first nine columns of (6.2) which does not vanish identically. The notation (6.2) is used for the determinant obtained by taking rows 1, $\cdots$, 10 and columns 5, 6, 7, 8, 9, 10, 16, 17, 18, 28 from Table I; a similar notation will be employed for a matrix in what follows. This ninth order minor, σ, which appears also in the matrix

$$\left\| \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 16 & 17 & 18 & 19 & 20 \end{matrix} \right\|, \tag{6.4}$$

can be bordered in two ways; let us now show that the resulting two determinants, namely

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 16 & 17 & 18 & 19 \end{vmatrix} \tag{6.5}$$

and

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 16 & 17 & 18 & 20 \end{vmatrix}, \tag{6.6}$$

vanish identically. Since (6.5) and (6.6) vanish if $\alpha = 0$, we can limit ourselves to the case $\alpha \neq 0$. Multiply columns 5, 6, 7, 8, 9, 10 in (6.5) and (6.6) through by certain quantities and add to the remaining columns; we then find that (6.5) and (6.6) are equal respectively to the determinants of Table II and Table III multiplied by $2\alpha^2$. Then multiply the first and second rows of Tables

$b\beta - ad$	$\beta c - ae$	$bd - a\gamma$	$cd - 2af + be$
$\alpha d - ab$	$\alpha e - ac$	$\alpha\gamma - b^2$	$2\alpha f - 2bc$
$\alpha\beta - a^2$		$\alpha d - ab$	$\alpha e - ac$
	$\alpha\beta - a^2$		$\alpha d - ab$

TABLE II

$b\beta - ad$	$\beta c - ae$	$bd - a\gamma$	$ce - a\delta$
$\alpha d - ab$	$\alpha e - ac$	$\alpha\gamma - b^2$	$\alpha\delta - c^2$
$\alpha\beta - a^2$		$\alpha d - ab$	
	$\alpha\beta - a^2$		$\alpha e - ac$

TABLE III

II and III by α and $-a$ respectively. Then if we multiply the third and fourth rows by b and c respectively and add to the second rows, we find in each case

[15] *Loc. cit.*, Thomas and Titt, p. 21.

that the first and second rows of the resulting table are identical. This completes the proof that (6.5) and (6.6) vanish identically.

Hence (6.4) is of rank nine[16] and $L_{12} = 3$ and $L_{22} = 1$. The left members of a regular system corresponding to (6.1) are therefore

$$\frac{\partial^2}{\partial x^1\, \partial x^1}\{g_{22}, g_{23}, g_{24}, g_{33}, g_{34}, g_{44}\}; \frac{\partial^2}{\partial x^1\, \partial x^2}\{g_{23}, g_{24}, g_{33}\}; \frac{\partial^2 g_{33}}{\partial x^2\, \partial x^2}.$$

The four identities required by Theorem II are given by the four divergence relations

$$R^{\beta}_{\alpha,\,\beta} - \tfrac{1}{2}\frac{\partial R}{\partial x^{\alpha}} = 0, \tag{6.7}$$

where

$$R = g^{\alpha\beta} R_{\alpha\beta}.^{17}$$

From (3.7) we have the following scheme for the arbitrary data:

$$g_{1\alpha}\ (\alpha = 1, 2, 3, 4) \sim \text{Completely Arbitrary}$$

$$g_{22}, g_{23}, g_{24}, g_{34}, g_{44}, \frac{\partial g_{22}}{\partial x^1}, \frac{\partial g_{34}}{\partial x^1}, \frac{\partial g_{44}}{\partial x^1} \sim F(x^2, x^3, x^4) \text{ over } x^1 = 0 \tag{6.8}$$

$$g_{33}, \frac{\partial g_{33}}{\partial x^2}, \frac{\partial g_{33}}{\partial x^1}, \frac{\partial g_{23}}{\partial x^1}\frac{\partial g_{24}}{\partial x^1} \sim F(x^3, x^4) \text{ over } x^1 = x^2 = 0.$$

EXISTENCE THEOREM A. *Let the arbitrary functions in* (6.8) *be assigned analytic in the neighborhood of* $x^{\alpha} = 0$, *such that the determinant* $|(g_{\alpha\beta})_0|$ *and the quantity* (6.3) *are different from zero at* $x^{\alpha} = 0$. *Then there exists one, and only one, set of functions* $g_{ij}(x)$ $(i, j = 2, 3, 4)$, *analytic in the neighborhood of* $x^{\alpha} = 0$, *which together with their first derivatives reduce to the assigned functions* (6.8) *over the surfaces indicated, and which together with* $g_{1\alpha}(x)$ *constitute a solution of* (6.1).

With the above example before us we are in a position to see why it is that in §2 we adopted the order $L_{11}, L_{12}, L_{22}, L_{13}, \cdots, L_{nn}$ for our regular form rather than the order $L_{11}, L_{12}, L_{13}, \cdots, L_{1n}, L_{22}, \cdots, L_{nn}$. With the latter ordering one might expect the numbers L_{11}, L_{12}, L_{13} for (6.1) would have the values 6, 3, 1 respectively. If this were possible there would be five conditions of integrability instead of four. However we have seen in §4 that there cannot exist more than four identities (4.9) for the system (6.1). This seems to indicate that the regular form (3.4) is unique; in fact we can show by direct calculation that the system (6.1) furnishes an example for which the second of the above orderings is impossible. For by continuing the argument that was used to show that the matrix (6.4) is of rank nine, we can show that the rank of the entire matrix of the coefficients of the derivatives $\partial^2 g_{\alpha\beta}/\partial x^1 \partial x^{\gamma}$ $(\alpha, \beta, \gamma = 1, 2, 3, 4)$ in the system

[16] *Loc. cit.*, Bocher, p. 54.

[17] L. P. Eisenhart, *Riemannian Geometry*, p. 82.

(6.1) is also nine. This calculation merely involves showing that three more determinants analogous to (6.5) and (6.6) vanish identically. Since it is impossible to solve the system (6.1) for ten derivatives $\partial^2 g_{\alpha\beta}/\partial x^1 \partial x^2$, a regular form analogous to (3.4) cannot be obtained with the second of the above orderings.[18]

The following theorem concerning solutions of Einstein's equations is of interest in that it seems to have no analogue in the theory of second order equations of classical physics. Consider any analytic solution $g_{\alpha\beta}(x)$ of (6.1) which for the sake of definiteness we will assume has signature -2. If the surface $x^1 = 0$ is not a characteristic surface, we can make a change of coordinates so that the geodesics orthogonal to the surface $x^1 = 0$ become the curves of parameter x^1, and x^1 measures the arc along these geodesics. In such a coordinate system the line element takes the form

$$(ds)^2 = (dx^1)^2 - g_{ij}dx^i dx^j \qquad (i, j = 2, 3, 4), \tag{6.9}$$

where $g_{ij}\,dx^i\,dx^j$ is a positive definite quadratic form.[19] Under transformations of the form

$$x^1 = \bar{x}^1, \qquad x^i = f^i(\bar{x}^2, \bar{x}^3, \bar{x}^4) \qquad (i = 2, 3, 4), \tag{6.10}$$

the conditions

$$g_{11} = 1, \qquad g_{1i} = 0 \tag{6.11}$$

are preserved.

Let us assume that the functions g_{ij} satisfy the conditions

$$\left(\frac{\partial g_{ij}}{\partial x^1}\right)_{x^1=0} = 0, \qquad \left(\frac{\partial^2 g_{ij}}{\partial x^1\,\partial x^1}\right)_{x^1=0} = 0. \tag{6.12}$$

The left members of (6.12) transform as the components of two second order tensors under the transformation (6.10); hence the conditions (6.12) are pre-

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[18] C. Lanczos, *Zur frage der regulären lösungen der Einsteinschen Gravitationsgleichungen*, Ann. der Phy., (5), 13, (1932), pp. 621–35, has considered boundary value problems for Einstein's gravitational equations of the initial value type. Although he does not obtain existence theorems showing the exact form of the arbitrariness as is done in §6, he arrives at the conclusion that the unknowns g_{ij} $(i, j = 2, 3, 4)$ and their first derivatives $\partial g_{ij}/\partial x^1$ can be taken arbitrarily over the surface $x^1 = 0$, subject to four restrictions. Thus far this agrees with the form of (6.8), but he concludes that the unknowns g_{ij} themselves will be completely arbitrary over the surface $x^1 = 0$. It can be shown that it is impossible to obtain an existence theorem for analytic solutions with this type of initial conditions. For it would mean that the complementary set of monomials corresponding to each of the unknowns g_{ij} must contain the monomial 1 with the multipliers x^2, x^3, x^4. From the definition of the complementary set N_1, we see that this would mean that all of the derivatives appearing on the left of the orthonomic system corresponding to (6.1) must contain at least one differentiation with respect to x^1. But the above results show that this is impossible.

[19] *Loc. cit.*, Eisenhart, p. 57.

served. Then put $x^1 = 0$ in (6.1). By (6.11) and (6.12), these ten equations reduce to the six equations

$$(R_{ij})_0 = 0, \tag{6.13}$$

where $(R_{ij})_0$ is used to denote the contracted Riemannian curvature tensor for the quadratic form

$$(g_{ij})_0\, dx^i dx^j, \tag{6.14}$$

where $(g_{ij})_0$ is used to denote $(g_{ij})_{x^1=0}$. It is well known that for the case $n = 3$, the equations (6.13) imply that

$$(R^i_{jkl})_0 = 0.$$

Hence it will be possible to find a coordinate system defined by a transformation of the form (6.10) in which

$$(g_{ij})_0 = \text{constant}. \tag{6.15}$$

However, the quantities (6.11), (6.12), and (6.15) completely determine the arbitrary data (6.8), i.e.,

$$g_{11} = 1, \qquad g_{1i} = 0,$$

$$\left.\begin{aligned} &g_{22}, g_{23}, g_{24}, g_{34}, g_{44} = \text{constant} \\ &\frac{\partial g_{22}}{\partial x^1} = \frac{\partial g_{34}}{\partial x^1} = \frac{\partial g_{44}}{\partial x^1} = 0 \end{aligned}\right\} \quad \text{over } x^1 = 0, \tag{6.16}$$

$$g_{33} = \text{constant}, \frac{\partial g_{33}}{\partial x^2} = \frac{\partial g_{33}}{\partial x^1} = \frac{\partial g_{23}}{\partial x^1} = \frac{\partial g_{24}}{\partial x^1} = 0 \text{ over } x^1 = x^2 = 0.$$

Since $g_{\alpha\beta} = \text{constant}$ satisfy both the equations (6.1) and the initial conditions (6.16), we can state the following.

THEOREM. *The only analytic solution of* (6.1) *satisfying the conditions* (6.12) *is the Euclidean solution* $g_{\alpha\beta} = \text{constant}$.

Let us now proceed to the study of the characteristic surfaces of the system (6.1). From Table I it is seen that g^{11} must vanish if the rank of the matrix of the coefficients of the derivatives $\partial^2 g_{\alpha\beta}/\partial x^1 \partial x^1$ in the system (6.1) is to be less than L_{11}. Hence from the definition in §5, it follows that a surface $\Phi = 0$ over which

$$g^{\alpha\beta}\Phi_\alpha\Phi_\beta = 0 \tag{6.17}$$

is a three dimensional characteristic surface. In fact, if we make a transformation of the form (5.2), the left member of (6.17) becomes $\bar{g}^{11}$. Thus, if we know a solution $g_{\alpha\beta}(x)$ of the system (6.1), the condition (6.17) constitutes a first order partial differential equation for the determination of the unknown Φ such that $\Phi = 0$ will be a characteristic surface. The first order equation (6.17) will have characteristic curves given by the equations

$$\frac{dx^1}{2g^{1\alpha}\Phi_\alpha} = \frac{dx^2}{2g^{2\alpha}\Phi_\alpha} = \frac{dx^3}{2g^{3\alpha}\Phi_\alpha} = \frac{dx^4}{2g^{4\alpha}\Phi_\alpha} = ds, \tag{6.18}$$

which are called the *bicharacteristics* of the system (6.1).[20]

In §5 we saw that the general existence theorem failed to apply when the surface $x^1 = 0$, bearing a portion of the arbitrary data, is a characteristic surface. Let us now proceed to show that Cauchy's problem fails by becoming indeterminate in this case. Hence suppose that $g^{11} = 0$ over $x^1 = 0$ for some analytic solution $g_{\alpha\beta}(x)$ of (6.1), i.e., that the surface $\Phi = x^1 = 0$ is a characteristic surface for the solution $g_{\alpha\beta}$. Then select the surface $x^2 = 0$ so that the surface $x^1 = x^2 = 0$ is not tangent to a bicharacteristic at $x^\alpha = 0$. From (6.18) we see that this latter condition implies that g^{12} is different from zero at $x^\alpha = 0$. Then select the surface $x^3 = 0$ so that the determinant from Table I,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 16 & 17 & 18 & 19 & 20 & 28 \end{vmatrix} = 8a^4\,(\beta b^2 - 2abd + a^2\gamma)^2 \tag{6.19}$$

at $x^\alpha = 0$ (disregarding algebraic sign), does not vanish at $x^\alpha = 0$.

Because of the non-vanishing of (6.19) at $x^\alpha = 0$, we can put the system (6.1) in the form

$$\frac{\partial^2}{\partial x^1 \partial x^1}\begin{Bmatrix} g_{22} \\ g_{23} \\ g_{24} \\ g_{33} \end{Bmatrix} = W\frac{\partial^2 g_{34}}{\partial x^1 \partial x^1} + W\frac{\partial^2 g_{44}}{\partial x^1 \partial x^1} + \Sigma\frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\tau} + \star$$

$$(\text{if } \sigma = \tau,\ \sigma \neq 1),$$

$$\frac{\partial^2}{\partial x^1 \partial x^2}\begin{Bmatrix} g_{23} \\ g_{24} \\ g_{33} \\ g_{34} \\ g_{44} \end{Bmatrix} = \alpha W\frac{\partial^2 g_{34}}{\partial x^1 \partial x^1} + \alpha W\frac{\partial^2 g_{44}}{\partial x^1 \partial x^1} + \Sigma\frac{\partial^2 g_{\mu\nu}}{\partial x^\epsilon \partial x^\delta} + \star \tag{6.20}$$

$$(\epsilon \leqq \delta;\ \text{if } \epsilon = 1,\ \delta \neq 1, 2),$$

$$\frac{\partial^2 g_{33}}{\partial x^2 \partial x^2} = \alpha W\frac{\partial^2 g_{34}}{\partial x^1 \partial x^1} + \alpha W\frac{\partial^2 g_{44}}{\partial x^1 \partial x^1} + W\frac{\partial^2 g_{34}}{\partial x^2 \partial x^2} + W\frac{\partial^2 g_{44}}{\partial x^2 \partial x^2}$$

$$+ \Sigma\frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\gamma} + \star \quad (\lambda \leqq \gamma;\ \text{if } \lambda = 1, 2;\ \gamma \neq 1, 2),$$

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[20] E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, (1921), p. 184. Cf. *loc. cit.*, Hadamard.

where W denotes various rational functions of $g^{\alpha\beta}$ and Σ denotes a sum of second derivatives with coefficients W. In addition, the condition

$$g^{11} = 0 \quad \text{over } x^1 = 0,$$

which implies that

$$g^{1j}\, g^{1k}\, \frac{\partial g_{jk}}{\partial x^i} = 0 \text{ over } x^1 = 0 \qquad (i, j, k = 2, 3, 4), \tag{6.21}$$

must be satisfied. Since $g^{12} \neq 0$ at $x^i = 0$, the system (6.21) can be replaced by

$$\frac{\partial g_{22}}{\partial x^i} = \Sigma\, \frac{\partial g_{jk}}{\partial x^i} \qquad \begin{pmatrix} i, j, k = 2, 3, 4 \\ \text{if } j = k,\ j \neq 2 \end{pmatrix}. \tag{6.22}$$

We shall adopt the assignment of marks given by (3.2), and in addition, assign third marks to the unknowns as follows:

(6.23)

Unknown	g_{22}	g_{23}, g_{24}, g_{33}	Other Unknowns
Third Mark	2	1	0

Upon examination of the conditions for a normal system of differential equations, it is seen that the combined system (6.20) and (6.22), in which (6.20) constitutes the system R and (6.22) the system S, will be *normal* provided that we can show that the set of monomials corresponding to each unknown is *strongly complete* Taking account of §3, we need examine only the sets for g_{34}, g_{44}, and g_{22}.

(6.24)

	Set	Multipliers	Non-mult.	Comp. Set	Multipliers
	$x_1\, x_2$	x_1, x_2, x_3, x_4	$\cdots$	$N_1 \sim 1$	x_2, x_3, x_4
				$N_2 \sim x_1$	x_1, x_3, x_4
$L \sim x_1\, x_1$		x_1, x_2, x_3, x_4	$\cdots$	$N_1 \sim x_1$	x_2, x_3, x_4
K	x_2	x_2, x_3, x_4	$\cdots$	$N_4 \sim 1$	$\cdots$
K	x_3	x_3, x_4	x_2		
K	x_4	x_4	x_2, x_3		

Examination of the table (6.24) shows that the combined system (6.20) and (6.22) is normal and the complete integrability will be demonstrated in §7. The arbitrary functions are exhibited in the following scheme:

$$\begin{aligned} &g_{1\alpha}\ (\alpha = 1, 2, 3, 4) && \sim \text{Completely Arbitrary} \\ &g_{23}, g_{24}, g_{34}, g_{44}, \frac{\partial g_{22}}{\partial x^1} && \sim F(x^2, x^3, x^4) \text{ over } x^1 = 0 \\ &\frac{\partial g_{34}}{\partial x^1}, \frac{\partial g_{44}}{\partial x^1} && \sim F(x^1, x^3, x^4) \text{ over } x^2 = 0 \end{aligned} \tag{6.25}$$

$$g_{33},\ \frac{\partial g_{33}}{\partial x^2},\ \frac{\partial g_{23}}{\partial x^1},\ \frac{\partial g_{24}}{\partial x^1},\ \frac{\partial g_{33}}{\partial x^1} \quad \sim F(x^3, x^4) \text{ over } x^1 = x^2 = 0$$

$$(g_{22})_0 \quad \sim \text{Arbitrary.}$$

EXISTENCE THEOREM B. *Let the arbitrary functions in* (6.25) *be assigned analytic in the neighborhood of* $x^\alpha = 0$, (1) *such that the determinant* $|(g_{\alpha\beta})_0|$ *is non-singular,* (2) *such that* $(g^{11})_0$ *vanishes and* (6.19) *is different from zero at* $x^\alpha = 0$. *Then there exists one, and only one, set of functions* $g_{ij}(x)$ $(i, j = 2, 3, 4)$ *analytic in the neighborhood of* $x^\alpha = 0$ *which, together with their first derivatives, reduce to the assigned functions in* (6.25) *over the indicated surfaces and which, together with* $g_{1\alpha}(x)$, *constitute a solution of* (6.1) *for which* $x^1 = 0$ *is a characteristic surface.*

Cauchy's problem for the system (6.1) is the problem of determining the unknowns g_{ij} $(i, j = 2, 3, 4)$ from the knowledge of the values of these unknowns and their first derivatives $\partial g_{ij}/\partial x^1$ over the surface $x^1 = 0$. Let us proceed to show that Cauchy's problem for a characteristic is indeterminate. In order to do this suppose that the arbitrary functions in (6.25) corresponding to $\partial g_{34}/\partial x^1$ and $\partial g_{44}/\partial x^1$ are of the form

$$\Psi = (x^1)^3\, \mu_3(x) + (x^1)^4\, \mu_4(x) + \cdots,$$

where $\mu_\sigma(x)$ is a function of x^3 and x^4 alone. In order to calculate from (6.20) and (6.22) the unknowns g_{ij} and their first derivatives $\partial g_{ij}/\partial x^1$ over $x^1 = 0$, we must know at $x^\alpha = 0$ all the derivatives of g_{ij} containing not more than one differentiation with respect to x^1. From the form of (6.20) and (6.22), it is seen that no differentiations of the equations with respect to x^1 will be required in order to calculate these derivatives. On account of the form of the function Ψ the values of these derivatives will be independent of the choice of the functions $\mu_\sigma(x)$. Hence if the surface $x^1 = 0$ is a characteristic surface, there is an infinite number of sets of functions $g_{ij}(x)$ such that the different sets of functions g_{ij} and their first derivatives $\partial g_{ij}/\partial x^1$ take on the same values over $x^1 = 0$. Thus Cauchy's problem is indeterminate.

Since L_{11}, L_{12}, and L_{22} are the only numbers $L_{\alpha\beta}$ which are not zero, it follows from the definition in §5 that the system (6.1) has no one dimensional characteristics. This result was not obtained in the former treatment. On account of the fact that, as yet, we are unable to give any physical interpretation of the two dimensional characteristic surfaces, a discussion of them will not be given here.

7. **Complete Integrability of a Normal System.** The demonstration of the complete integrability of the combined system (6.20) and (6.22) will be given especial consideration because the method to be used is capable of quite general application. The demonstration utilizes the general ideas of the sufficiency proof of §4. That the method applies also to the more general problem depends on the fact that the former proof did not essentially make use of the form of the left members of the system (3.4).

We need consider only the equations (6.20) in testing for complete integrability. For, recall the condition for strong completeness of a set M of monomials, namely, that each product of a monomial $\bar{K}$ by one of its non-multipliers belongs to the *class* ($\mathfrak{M}$) and that each product of a monomial $\bar{L}$ by one of its non-multipliers belong to the class ($\mathfrak{L}$). There is then one integrability condition corresponding to each product of a monomial in the set K of (6.24) by one of its non-multipliers. But since each of these products belongs to the class ($\mathfrak{L}$), the equations (6.20) are not involved. We know *a priori* that the three integrability conditions thus obtained are satisfied since the system (6.21) was obtained by differentiating a known algebraic condition $g^{11} = 0$. The remaining four integrability conditions involve the system (6.20) alone.

It is also necessary to notice that the four indentities (6.7) are linearly independent in the derivatives $\partial R_{\alpha\beta}/\partial x^\gamma$ over a characteristic surface $x^1 = 0$, i.e., a surface $x^1 = 0$ over which g^{11} vanishes, under the conditions of Existence Theorem B. In particular, the independence depends on the non-vanishing of the quantity (6.19) at $x^\alpha = 0$, since the determinant of the coefficients of the derivatives of R_{22}, R_{23}, R_{24}, R_{33} with respect to x^1 in the system (6.7) is

$$\tfrac{1}{2}\, a\,(a^2\gamma - 2abd + b^2\beta).$$

Corresponding to the notation used in (4.9), the identities (6.7) can be replaced by the equations

$$\Lambda_a^{\alpha\beta\gamma}\frac{\partial R_{\alpha\beta}}{\partial x^\gamma} = 0 \qquad (a = 1, \cdots, 4), \tag{7.1}$$

which are satisfied in consequence of (6.1). Let the result of transposing all terms in (6.20) to the left hand members be denoted by

$$X_i = 0 \qquad (i = 1, \cdots, 10) \tag{7.2}$$

corresponding to (3.4). The equations

$$Y_{i\alpha} = 0 \qquad \begin{pmatrix} i = 1, \cdots, 10 \\ \alpha = 1, \cdots, 4 \end{pmatrix} \tag{7.3}$$

will differ from the corresponding equations (4.5) in the important respect that some of the right hand derivatives with coefficients which contain α as a factor will *follow* their corresponding left members under the ordering given by (3.2) and (6.23). There will be four conditions corresponding to (4.8).

Aside from the above remarks the proof of the complete integrability of the normal system (6.20) and (6.22) will be formally the same as the sufficiency proof in §4.

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BOOLEAN LINEAR ASSOCIATIVE ALGEBRA

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1. **Introduction.** The theory of linear algebra in which the coefficients are elements of a Boolean algebra or field, hereafter denoted by $\mathfrak{B}$, has hitherto been confined in the main to the theory of relatives that is, to matrices whose coordinates are either 0 or 1. A number of peculiar difficulties are met with in classifying the types of such algebras and it has not proved possible as yet to give a complete classification but only to provide means for simplifying the task of enumeration. The relative complexity of the problem appears in the fact that there are 25 distinct irreducible types in two units.

2. **The general algebra.** Two hypernumbers $x = (\xi_1, \xi_2, \cdots, \xi_n)$ $y = (\eta_1, \eta_2, \cdots, \eta_n)$ with coordinates in $\mathfrak{B}$ are equal if, and only if, $\xi_i = \eta_i$ and as usual if u_i is the hypernumber with 1 in the i-th place and 0 elsewhere, then $x = \Sigma\xi_i u_i$. The sum $z = x + y$ is defined as $\zeta_i = \xi_i + \eta_i$. Multiplication, which is assumed associative and distributive, is defined by

$$u_i u_j = \Sigma\gamma_{ijk}u_k, \qquad \Sigma\gamma_{ijp}\gamma_{pkq} = \Sigma\gamma_{jkp}\gamma_{ipq}.$$

The minimum Boolean field in which this algebra can exist is then the one which is generated by the γ_{ijk} under addition and multiplication. This field is finite, say $\sum_1^m \alpha_i = 1$, $\alpha_i\alpha_j = 0$ $(i \neq j)$ where $\alpha\gamma_{ijk}$ is either 0 or α_p for every choice of subscripts. Hence the algebra A can be expressed as the direct sum of m algebras A_i in which A_i has the universe α_i and the units $u_{ij} \equiv \alpha_i u_j$; and in A_i the γ's are either 0 or 1. In classifying algebras we may therefore restrict ourselves to this case. As an illustration we may take the algebra of four units $e_{ij}(i, j = 1, 2)$ where

$$e_{ii}^2 = e_{ii}, \qquad e_{ii}e_{ij} = e_{ij} = e_{ij}\,e_{jj}, \qquad e_{ij}e_{ji} = \alpha e_{ii}, \qquad e_{ij}e_{pq} = 0 \ (j \neq p).$$

This is the direct sum of an algebra with the universe α, which amounts to putting $\alpha = 1$, and an algebra with the universe $\bar{\alpha}$, in which $e_{ij}e_{ji} = 0$.

3. **Bases.** One reason for the large number of types mentioned in the introduction is the limited choice of basic elements. If $v_i(i = 1, 2, \cdots, n)$ is a second basis of A, we must have $v_i = \sum_j p_{ij}u_j$, or say

$$(v_i) = P(u_i), \qquad P = \| p_{ij} \|$$

and simultaneously $u_i = \Sigma q_{ij} v_j$ or

$$(u_i) = Q(v_i);$$

and if these bases are equivalent, they must contain the same number of non-zero elements, and

$$PQ = 1 = QP. \tag{3.1}$$

If P is given, it is a necessary and sufficient condition for the existence of Q that

$$\sum_j p_{ij} = 1 = \sum_i p_{ij}, \qquad p_{ij}p_{ik} = 0 = p_{ji}p_{ki} \ (j \neq k) \tag{3.2}$$

so that these matrices are a generalization of the Latin square.

If the universe $[p_{ij}]$ has the units π_i $(i = 1, 2, \cdots, m)$, then $P = \Sigma \pi_i P = \Sigma \pi_i P_i$ and each matrix P_i has one 1 in each row and column and 0 elsewhere, and so corresponds to a permutation; there are $n!$ such matrices and the product of the associated π's for any two is 0. Hence the general solution of (3.1) is $\Sigma \pi_i P_i$ where $\Sigma \pi_i = 1$, $\pi_i \pi_j = 0$ $(i \neq j)$ and the P_i are the $n!$ matrices corresponding to the permutations of $1, 2, \cdots, n$.

The condition (3.2) may also be stated as follows. We first define the determinant of P as $\Sigma p_{1i_1} p_{2i_2} \cdots p_{ni_n} q_{(i)}$ where $q_{(i)}$ is the product of the complements of the elements which do not appear in the product associated with it and the sum extends over all combinations of the subscripts. The condition (3.2) is equivalent to

$$|P| = 1. \tag{3.3}$$

This determinant has the property that, if P and Q are any two matrices, then $|PQ| = |P||Q|$.

An important consequence of the limitations on the form of P is that an element $x = \Sigma \xi_i u_i$ can form an element of a basis if, and only if, $\Sigma \xi_i = 1$, $\xi_i \xi_j = 0$ $(i \neq j)$. We shall say x is *basic* if it can form an element of a basis. If $x_1, x_2, \cdots, x_r$ are given at random, the properties of the set $\Sigma \alpha_i x_i$ require further study, for the apparent lack of simple results handicaps the study of subalgebras of a given algebra.

4. **Irreducible algebras with an identity element.** Let u_i $(i = 1, \cdots, n)$ be a basis of A and $e = \Sigma \xi_i u_i$ the identity element. If say $\xi_1 \neq 0$, we may take the algebra modulo ξ_1 which makes the coefficient of u_1 equal 1, and either gives a reduced part of A or, if of the same type as A, gives a new basis in which $\xi_1 = 1$. Since A is assumed irreducible, we may therefore choose a basis

$$e_i \ (i = 1, 2, \cdots, r), \qquad u_i \ (i = 1, 2, \cdots, s), \qquad e = \sum_1^r e_i. \tag{4.1}$$

If we set

$$e_i e_j = \Sigma \gamma_{ijk} e_k + \Sigma \delta_{ijk} u_k,$$

then $e_i = ee_i = e_ie$ gives $\gamma_{ijk} = 0$ $(k \neq i, j)$, $\delta_{ijk} = 0$ and this together with $e^2 = e$ gives

$$e_i^2 = e_i, \qquad e_ie_j = 0 \ (i \neq j). \tag{4.2}$$

If similarly

$$e_iu_j = \Sigma\alpha_{ijp}e_p + \Sigma\beta_{ijq}u_q,$$

we find $e_iu_j = a_{ij}u_j$, $a_{ij}a_{kj} = 0$ $(i \neq k)$; and in the same way $u_je_i = b_{ij}u_j$, $b_{ij}b_{kj} = 0$ $(i \neq k)$ and, since A is irreducible, $a_{ij} = 0, 1$; $b_{ij} = 0, 1$. We therefore get a Peirce decomposition of the basis. It should be noted that e cannot form an element of any basis if $r > 1$.

If by a different choice of basis we get $e = \sum_1^s v_i$, $v_i^2 = v_i$, $v_iv_j = 0$, then

$$e_iv_j = \Sigma\xi_{ijk}e_k + \Sigma\xi_{ijk}u_k.$$

But $\sum_j e_iv_j = e_i$. Hence $\xi_{ijk} = 0$ $(k \neq i)$, $\xi_{iji} = 0$ so that $e_iv_j = \xi_{iji}e_i \equiv \xi_{ij}e_i$, and summing with regard to i we have $v_j = \Sigma\xi_{ij}e_i$. Similarly $e_i = \Sigma\eta_{ij}v_j$ and hence $r = s$.

5. **Matric representation of an algebra which contains an identity.** If u_i $(i = 1, 2, \cdots, n)$ is a basis of A and x, y any elements we may set

$$u_ix = \Sigma\xi_{ij}u_j, \qquad u_iy = \Sigma\eta_{ij}u_j$$

and then

$$u_ixy = \Sigma\xi_{ij}\eta_{jk}u_k, \qquad u_i(x + y) = \Sigma(\xi_{ij} + \eta_{ij})u_j.$$

If $X = \|\xi_{ij}\| \sim x$, $Y = \|\eta_{ij}\| \sim y$, then $(u_i)x = X(u_i)$ and $X + Y \sim x + y$, $XY \sim xy$; also, if α is a scalar, $\alpha X \sim \alpha x$. Since A has an identity, $X = Y$ implies $x = y$ and conversely so that the set of matrices is simply isomorphic with the set of elements of A and so gives a representation of A. If there is no identity, the representation is not necessarily simply isomorphic.

If we denote the unit matrices by e_{ij} as usual and for brevity write x for X, we may set $x = \Sigma\xi_{ij}e_{ij}$. Suppose as before that $e = \sum_1^r e_i$, $e_i^2 = e_i$, $e_ie_j = 0$ where the e_i form a part of a basis of A. Setting

$$e_i = \Sigma\eta^i_{pq}e_{pq}, \qquad \Sigma e_i = e = \Sigma e_{jj},$$

we have $\eta^i_{pq} = 0$ $(p \neq q)$ and we may set

$$e_i = \sum_j \eta_{ij}e_{jj}, \qquad \sum_i \eta_{ij} = 1, \qquad \eta_{ij}\eta_{kj} = 0 \ (i \neq k). \tag{5.1}$$

Also, since

$$\left(\sum_j \eta_{ij}\right) e_i = (\Sigma\eta_{ij})\ \Sigma\eta_{ik}e_{kk} = \Sigma\eta_{ik}e_{kk} = e_i,$$

we must have $\Sigma\eta_{ij} = 1$.

6. **The algebra generated by a single element.** Let $x = \Sigma\xi_i u_i$ be an element of an algebra A to which an identity e has been added if necessary, and let $e = \sum_1^s u_i$. If $\alpha_0 = \Pi\bar{\xi}_i$, then $\alpha_0 x = 0$; and if $\lambda x = 0$, then $\lambda = \lambda\alpha_0$. The algebra $\{x\}$ generated by x is then in effect an algebra with the universe $\bar{\alpha}_0$. Again, let

$$\alpha_1 = \prod_1^s \xi_i \prod_{s+1}^n \bar{\xi}_i = \alpha_1\bar{\alpha}_0, \qquad \delta_1 = \bar{\alpha}_1\bar{\alpha}_0, \qquad \alpha_0 + \alpha_1 + \delta_1 = 1;$$

then $\alpha_1 x = \alpha_1$ and, if λx is a scalar, we have $\lambda\bar{\alpha}_0 = \lambda\alpha_1$. If we set

$$x = x_1 + w_1, \qquad x_1 = \alpha_1 x, \qquad w_1 = \delta_1 x, \qquad x_1 w_1 = 0 = w_1 x_1,$$

it is seen that $\{x\}$ is the direct sum of the Boolean algebra $\{x_1\} = \{\alpha_1\}$ with the universe α_1 and the algebra $\{w_1\}$ with the universe δ_1; and λw_1 equal to a scalar implies $\lambda\delta_1 = 0$.

We now treat w_1^2 in the same way; and so on. Finally we derive a succession of scalars

$$\alpha_0, \alpha_1, \cdots, \alpha_r, \delta_r, \qquad 1 = \Sigma\alpha_i + \delta_r, \qquad \alpha_i\alpha_j = 0 = \alpha_i\delta_r\ (i \neq j)$$

$$\alpha_0 x = 0$$

$$x = \sum_1^r x_i + w_r, \qquad x_i = \alpha_i x, \qquad w_r = \delta_r x \tag{6.1}$$

$$x_i^p \neq x_i^q\ (q < p \leq i), \qquad x_i^i = \alpha_i$$

and such that $\lambda w_r^p = 0$ implies either $w_r^p = 0$ or $\lambda\delta_r = 0$; δ_r and any or all of the α's may be 0 subject to the condition $1 = \Sigma\alpha_i + \delta_r$.

We say that x has an inverse with respect to a scalar α if there exists a number y of the algebra such $xy = \alpha = yx$. If we write

$$N(x) = \alpha_1 + \alpha_2 + \cdots + \alpha_r. \tag{6.2}$$

It is readily shown that $N(xy) = N(x)N(y)$; also, if x is a matrix, $N(x)$ is the determinant of x. The condition that x is not singular is then $N(x) \neq 0$. For, if this is so we have an element

$$y = \sum_1^r \alpha_i x_i^{i-1} \qquad (x_1^0 = \alpha_1) \tag{6.3}$$

for which $xy = N(x) = yx$.

The element w_r (6.1) can be normalized still further. Since the Boolean algebra generated by the coordinates of x and the γ_{ijk} is finite, only a finite number of powers of x can be distinct and hence there is a pair of integers p, q such that $x^{p+q} = x^p$ but $x^{i+j} \neq x^i$ for $i < p$ or $i = p, j < q$. We can therefore express w_r in the form

$$w_r = \Sigma x_{pq} + \Sigma z_p \tag{6.4}$$

where any terms which are not 0 have the properties

$$\begin{gathered} x_{pq}^{p+q} = x_{pq}^{p}, \qquad x_{pq}^{i+j} \neq x_{pq}^{i} \ (i < p;\ i = p,\ j < q) \\ z_p^p = 0, \qquad z_p^i \neq 0 \ (i < p) \end{gathered} \tag{6.5}$$

and associated with these there is a set of scalars β_{pq}, γ_p such that the product of any two is 0 and, if δ_r is the universe of w_r, then

$$\delta_r = \Sigma\beta_{pq} + \Sigma\gamma_p, \qquad x_{pq} = \beta_{pq}w_r, \qquad z_p = \gamma_p w_r. \tag{6.6}$$

Any of these scalars may be 0 subject to the condition on their sum.

7. **Algebras in which the identity is basic.** We shall assume that the algebra A is irreducible; let the basis be $e = u_1, u_2, \cdots, u_n$ the identity being e. If any u_i $(i > 1)$ is normalized as in (6.1) and (6.4), only one of the terms can be different from 0 since u_i is basic; and since e is basic, $\alpha_1 = 0$, that is, we have three types of elements in the basis, *(i)* $u_i^{p_i} = \alpha_i \neq 0$ *(ii)* $u_i^{p_i+q_i} = u_i^{p_i} \neq 0$ *(iii)* $u_i^{p_i} = 0$.

Suppose that u_i is of type *(i)*. We may take $\alpha_i = 1$ since otherwise by the same argument as in §3 we can regard A as the direct sum of an algebra in which $u_i^{p_i} = u_1$ and another in which it is 0. If $u_i^{p_i-1} = \Sigma\mu_j u_j$, then $u_1 = \Sigma\mu_j u_i u_j \neq 0$; for each $\mu_j u_i u_j$ which is not 0, we have $\mu_j u_i u_j = \mu_j u_1$; but $u_i u_j = \Sigma\gamma_{ijk}u_k$ in which each γ is 0 or 1; and hence $u_i u_j = u_1$. We cannot also have $u_i u_k = u_1$ since then $u_k = u_i^{p_i-1} = u_j$; hence $u_{r_i} = u_i^{p_i-1}$ is an element of the basis.

Again we may set $u_i^{p_i-2} = \Sigma\nu_j u_j$ so that $u_i^{p_i-1} = \Sigma\nu_j u_i u_j$ and the same argument as before shows that there is only one term and that the corresponding ν is 1. A repetition of the argument then shows that each u_i^k $(k = 1, 2, \cdots, p_i - 1)$ is an element of the basis. Further, if u_i and u_j are any elements of type *(i)* in the basis and we set $u_i u_j = \Sigma\gamma_s u_s$, then $u_j = \Sigma\gamma_s u_i^{p_i-1}u_s$ and as above we get only one term in the summation so that $u_i u_j = u_k$, say. Since $u_k u_j^{p_j-1}u_i^{p_i-1} = u_1$, the element u_k is of type *(i)*; hence in an irreducible algebra in which e is basic, the elements of the basis which are not singular form a group.

It is easily seen in the same way that the product of an element of type *(i)* into an element of type *(ii)* or *(iii)* is an element of type *(ii)* or *(iii)*. The example e, u, v, uv $(u^2 = e, uv = vu, v^2 = u + v)$ in which $v^5 = v^4$ shows that the classification of the remaining elements does not proceed along the same lines.

8. **Group algebras; primitive algebras.** We have seen in the previous section that, if $u_1, u_2, \cdots, u_n$ form a group, they may be taken as a basis of an

algebra in which every basic element is of type (i). Such an algebra is irreducible. For if A is any algebra in which the identity u_1 is basic and we suppose it reducible, we may set

$$u_i = \Sigma\alpha_{ij}v_j + \Sigma\beta_{ij}w_j, \qquad v_pv_q = 0 = w_pv_q$$

which reduces to $u_i = \alpha_iu_i + \bar{\alpha}_iu_i$. It follows from $u_1u_i = u_i$, $v_pw_q = 0$, that $\alpha_1\alpha_i = \alpha_i, \bar{\alpha}_1\alpha_i = 0$ and therefore $\alpha_i = \alpha_1$ for all i and the algebra is not properly reducible.

If in addition to the identity being basic every element of a basis has an inverse relative to that basis, then A is a group algebra. In general the element $x_1 = \Sigma(\xi_i\prod_{i\neq j}\bar{\xi}_j)u_i$ is an inverse of x provided it is not 0 which is the case if, and only if, $x = \sum_{i\neq j}\eta_{ij}(u_i + u_j)$. The general situation can be made clearer as follows. Let $u_{ri}(i = 1, 2, \cdots, \binom{n}{r}; r = 1, 2, \cdots, n)$ be the sum of the elements u_i taken r at a time. If $x = \Sigma\xi_iu_i$ in any algebra and γ_{ri} are the units of the subalgebra of $\mathfrak{B}$ generated by the ξ's so that $\gamma_{ri}\gamma_{sj} = 0$ for $(r, i) \neq (s, j)$, then the notation for the γ's can be so chosen that $x = \Sigma\gamma_{ri}u_{ri}$. If x has an inverse x_1, $x_1x = \alpha$, we may set $x_1 = \Sigma\gamma_{ri}v_{ri}$, whence $\gamma_{ri}u_{ri}v_{ri} = \alpha\gamma_{ri}u_1$. If the $\gamma_{ri} \neq 0$, we must have $u_{ri}v_{ri} = u_1$ (in an irreducible algebra). If A is a group algebra, it is readily shown that u_{ri} does not have any inverse except when $r = 1$, so that $x_1 = \alpha\Sigma\gamma_{1i}u_i$. This gives the result stated above and also shows that $xx_1 = u_1$ if, and only if, x is basic. A group algebra, then is characterized by the property that every basic element has an inverse and that only such elements and scalar multiples of them have inverses.

Let $A = (a_1, \cdots, a_\alpha)$ be an irreducible group algebra and $B = (b_1, \cdots, b_\beta)$ an invariant subalgebra, and let $A = \sum_1^r Ba_i$ be a complete set of co-sets of B in A; then, if $b = b_1 + b_2 + \cdots + b_\beta$, the elements ba_i $(i = 1, 2, \cdots, r)$ form a group algebra C in which the group is the quotient group A/B; further if x is an element of A, then bx is an element of C. It is worth noticing perhaps that, if a is the sum of the elements of the basis of A and bc the sum of those of C, then $a = bc = cb$, so that a theory of factors of a can be made to correspond to the theory of invariant subgroups of A.

The notion of invariant subalgebra can be generalized as follows. Let B be a subalgebra of A and $b_1, \cdots, b_k$ a basis of B; then $b = \sum_1^k b_i$ is independent of the basis chosen. We say that B is an invariant subalgebra of the first kind in A if (i) $bx = xb$ for every x in A (ii) there exist elements $v_1, \cdots, v_s$ of A such that every number bx can be expressed uniquely in the form $\sum_1^s \alpha_ibv_i$. The elements bv_i then define the corresponding quotient algebra. There is another type of

invariant subalgebra corresponding to that used in ordinary linear associative algebra; there are, however some difficulties connected with it in its most general form so that we shall give here only the simplest case. We say that $B = (b_1, b_2, \cdots, b_k)$ is an invariant subalgebra of second kind if (*i*) xb_i and b_ix are contained in B for every element b_i of the basis of B and for every element x of A and (*ii*) if the basis of B is a part of the basis of A. The difference algebra is then defined as in the usual theory.

There is another class of algebras which includes group algebras and has somewhat similar properties. With the same notation as before we set

$$U(x) = \Sigma\xi_i = \Sigma\gamma_{ri}, \qquad V(x) = \Sigma\xi_i \prod_{j \neq i} \bar{\xi}_j = \Sigma\gamma_{1i}. \tag{8.1}$$

$U(x)$ may be called the universe of x. In this notation the condition that x is basic is $U(x) = V(x) = 1$. If A is any irreducible algebra in which $u_iu_j \neq 0$ for any i, j, then

$$U(xy) = U(x)U(y), \qquad U(x + y) = U(x) + U(y). \tag{8.2}$$

Such algebras may be called primitive; they have the property that $xy = 0$ if, and only if, $U(x)U(y) = 0$ and so have many properties in common with $\mathfrak{B}$.

9. **Classification of algebras.** Owing largely to the difficulties which arise from the existence of non-basic elements, the general classification of the Boolean algebras cannot be carried as far as in the numerical theory. The Peirce classification of §4 can be extended somewhat when the algebras e_iAe_i are group algebras, but even in this case it appears from the examples given below that no obvious classification exists.

Let $u^k_{i\,j}$ $(i, j = 1, 2, \cdots, r; k = 1, 2, \cdots, r_{ij})$ be a basis in the Peirce form of an irreducible algebra A, the idempotent elements being $u^1_{i\,i} = e_i$. Let

$$u^i_{1\,1}u^p_{1\,2} = \Sigma\gamma_{ipq}u^q_{1\,2}, \qquad u^i_{1\,1}u^j_{1\,1} = e_1 \tag{9.1}$$

then

$$u^p_{1\,2} = \Sigma\gamma_{ipq}u^j_{1\,1}u^q_{1\,2}, \tag{9.2}$$

therefore for any i, q for which $\gamma_{ipq} \neq 0$ we must have $u^j_{1\,1}u^q_{1\,2} = u^p_{1\,2}$ or $u^i_{1\,1}u^p_{1\,2} = u^q_{1\,2}$, that is, the summation in (9.1) contains at most one term; all the elements $u^i_{1\,1}u^p_{1\,2}$ which are not 0 are therefore basic. It follows readily that for any i, p and k there is an r such that

$$u^i_{1\,1}u^p_{1\,2} = u^r_{1\,2}u^k_{2\,2}.$$

All the units $u^s_{1\,2}$ can therefore be put in the form

$$u^i_{1\,1}u^p_{1\,2} \tag{9.3}$$

where p takes on a limited number of values; and this can be done in such a way that $u^i_{1\,1}u^p_{1\,2} \neq u^j_{1\,1}u^q_{1\,2}$ unless possibly when $p = q$. Suppose now that

$$u^p_{1\,2}u^q_{2\,1} = \Sigma\delta_{pqs}u^s_{1\,1} \qquad (\delta_{pqs} = 0, 1)$$
$$u^p_{2\,1}u^q_{1\,2} = \Sigma\epsilon_{pqs}u^s_{2\,2}.$$

Then

$$u^p_{1\,2}u^q_{2\,1}u^r_{1\,2} = \Sigma\delta_{pqs}u^s_{1\,1}u^r_{1\,2} = \Sigma\epsilon_{qst}u^p_{1\,2}u^t_{2\,2}.$$

Hence for any p, q, s such that $\delta_{pqs} \neq 0$ we must have

$$u^s_{1\,1}u^r_{1\,2} = \Sigma\theta_{qst}u^p_{1\,2}u^t_{2\,2}$$

and therefore some t

$$u^s_{1\,1}u^r_{1\,2} = u^p_{1\,2}u^t_{2\,2}.$$

Here r is wholly arbitrary and p can be fixed unless every $\delta_{pqs} = 0$; it follows that only one value of r or p is necessary in (9.3).

Suppose now that every $u^p_{1\,2}u^q_{2\,1} = 0$; then from (9.3) also every $u^p_{2\,1}u^q_{1\,2} = 0$. It follows in a similar way that for all p, q, r

$$0 = u^p_{2\,1}u^q_{1\,r} = {}^p_{1\,2}u^q_{2\,r} = u^p_{r\,1}u^q_{1\,2} = u^p_{r\,2}u^q_{2\,2}$$

and hence $(u^p_{1\,2};\ u^q_{2\,1};\ p = 1, 2, \cdots, r_{12}, q = 1, 2, \cdots, r_{21})$ forms an invariant subalgebra of the second kind in A. Hence, if A has no invariant subalgebra of the second kind, we can find a set of elements $u_{ij}(i \neq j)$ such that every $u^p_{i\,j}$ can be put in either of the forms $u^r_{i\,i}u_{ij}, u_{ij}u^s_{j\,j}$. It should be noted that we may have $u^r_{i\,i}u_{ij} = u^t_{i\,i}u_{ij}$ even when $r \neq t$.

The following examples seem to show that the classification cannot be carried much farther by these means.

	e_{11}	u_{11}	e_{12}	u_{12}	e_{21}	u_{21}	e_{22}	u_{22}
e_{11}	e_{11}	u_{11}	e_{12}	u_{12}	0	0	0	0
u_{11}	u_{11}	e_{11}	e_{12}	u_{12}	0	0	0	0
e_{12}	0	0	0	0	v_{11}	w_{11}	e_{12}	u_{12}
u_{12}	0	0	0	0	w_{11}	v_{11}	u_{12}	e_{12}
e_{21}	e_{21}	u_{21}	v_{22}	w_{22}	0	0	0	0
u_{21}	u_{21}	e_{21}	w_{22}	v_{22}	0	0	0	0
e_{22}	0	0	0	0	e_{21}	u_{21}	e_{22}	u_{22}
u_{22}	0	0	0	0	e_{21}	u_{21}	u_{22}	e_{22}

If we put $v_{11} = w_{11} = e_{11} + u_{11}$, $v_{22} = w_{22} = e_{22} + u_{22}$, we get an algebra which does not seem to admit of further simplification. It has, however, a matric subalgebra of order 4, viz., $f_{ij} = e_{ij} + u_{ij}$ $(i, j = 1, 2)$ but cannot be expressed as the direct product of it and another algebra as in the case of the numerical theory. If $v_{ii} = e_{ii}$, $w_{ii} = u_{ii}$, the algebra is the direct product of the matric algebra (e_{ij}) of order 4 and the group algebra (e, u).

The following example shows that e_iAe_i may be a group algebra for some values of i and not for others.

	e_{11}	u_{11}	u_{12}	u_{21}	e_{22}	u_{22}
e_{11}	e_{11}	u_{11}	u_{12}	0	0	0
u_{11}	u_{11}	$e_{11}+u_{11}$	u_{12}	0	0	0
u_{12}	0	0	0	$e_{11}+u_{11}$	u_{12}	u_{12}
u_{21}	u_{21}	u_{21}	u_{22}	0	0	0
e_{22}	0	0	0	u_{21}	e_{22}	u_{22}
u_{22}	0	0	0	u_{21}	u_{22}	e_{22}

The complexity of the problem of classification is shown by the following list of irreducible algebras of order 2. If we set

$$u^2 = \lambda u + \mu v, \qquad v^2 = \xi u + \eta v, \qquad uv = \alpha u + \beta v, \qquad vu = \gamma u + \delta v$$

we readily find the following types (dots are used in place of 0's).

	λ	μ	ξ	η	α	β	γ	δ
1	.	.	.	1	.	.	1	.
2	.	.	.	1	1	.	.	.
3	.	.	.	1	1	.	1	.
4	.	.	1	.	.	.	.	.
5	.	.	1	1	.	.	.	.
6	.	.	1	1	.	.	1	.
7	.	.	1	1	1	.	.	.
8	.	.	1	1	1	.	1	.
9	.	1	.	1	.	1	.	1
10	.	1	.	1	1	.	1	.
11	.	1	1	1	1	1	1	1
12	1	.	.	1	.	1	.	1

	λ	μ	ξ	η	α	β	γ	δ
13	1	.	.	1	1	1	1	.
14	1	.	.	1	.	1	1	.
15	1	.	.	1	1	.	.	1
16	1	.	.	1	1	.	1	1
17	1	.	.	1	1	1	1	1
18	1	.	1	1	.	1	.	1
19	1	.	1	1	.	1	1	1
20	1	.	1	1	1	.	1	.
21	1	.	1	1	1	.	1	1
22	1	.	1	1	1	1	.	1
23	1	.	1	1	1	1	1	.
24	1	.	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1

Of these four have an identity—3, 10, 12, 18— and in six every element is idempotent, viz., 12 to 17; the only group algebra is 10. The commutative types are: 3, 4, 5, 8, 9, 10, 11, 12, 17, 18, 20, 24, 25. For types 9 to 25 $U(x) = \xi_1 + \xi_2$ has the properties given in (8.2). Another multiplicative norm is obtained by using the determinant in the matrix representation, which exists even when there is no identity; it gives 0, however, except in the following cases

$$2,\xi_2;\ 3,\ 13,\ \bar{\xi}_1\xi_2;\ 12,\ 18,\ 22\ \xi_1\bar{\xi}_2;\ 10\ \xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2.$$

10. **Applications and generalizations.** It seems likely that these algebras will have varied applications but it must suffice here to give three simple cases. In types 7 and 8 of the previous section we have

$$(7)\ (a, \bar{a})\ (c, \bar{c}) = (ac, \bar{}(ac)), \qquad (8)\ (a, \bar{a})\ (c, \bar{c}) = (a + c, \bar{}(a + c))$$

which have obvious interpretations in the theory of sets of points. Again, if in

$$x = a_3 + a_2 u + a_1 u^2 + a_0 u^3, \qquad u^4 = u^3,$$

we interpret a_i as an open i-cell whose boundary is a_{i-1} and if $y = \Sigma b_i u^{3-i}$ is a second number of the same form, then the successive coefficients in xy have the same interpretation, the common part of a_3 and b_3 being $a_3 b_3$, its boundary $a_3 b_2 + a_2 b_3$, and so on.

The definition of an algebra can be extended considerably by altering the definition of addition. For instance we may set

$$(a, b) + (c, d) = (a + c, bd), \qquad (a, b)\,(c, d) = (ac, b + d), \qquad 0 = (0, 1).$$

Again, instead of assuming the product to be linear in the coordinates of each factor, we may use a law such as

$$(a, b)\,(c, d) = (k(ac + ad) + \bar{k}(a + c + d),\ kb(c + d) + \bar{k}(b + c + d))$$

which corresponds to using $a \times b = kab + \bar{k}(a + b)$ in place of ordinary multiplication in $\mathfrak{B}$. This algebra is reducible into two parts with the universes k and $\bar{k}$; the first is of the type already considered, the second is

$$(a, b)\,(c, d) = (a + c + d, b + c + d).$$

It should be noticed however that we cannot now write $m(a, b) = (ma, mb)$ which restricts the type of algebraic operation available in using the algebra.

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CRITERIA FOR THE IRREDUCIBILITY OF POLYNOMIALS

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This note contains a few remarks to our paper with the same title which appeared in these Annals, **34** (1933) pp. 81–99.

Dr. Spitz in Budapest informs us, that he has checked the list of exceptions to theorem 4, p. 86 and has found, that for $n = 3$ the case

$$F(x) = -x^2 - x + 1,\ P(x) = (x + 1)\,x(x - 2),$$
$$F(P(x)) = [(x + 1)\,(x - 2) + 1]\,[(x + 1)\,(x - 2)\,(-x^2 + 1) + 1]$$

must be added, while for $n = 2$ the list should read

$$F(x) = -a^2x^2 + 4x + 1,\ P(x) = x(x - 1)$$
$$F(P(x)) = [aP(x) - 2x + 1]\,[-aP(x) - 2x + 1]$$
$$F(x) = -a^2x^2 + x + 1,\ P(x) = x(x - 2),$$
$$F(P(x)) = [aP(x) - x + 1]\,[-aP(x) - x + 1].$$

Dr. Spitz has also given a complete list of exceptions for the corresponding theorem in the quadratic imaginary field $K(i)$.

In the comments to theorem 16, p. 93, line 4 from below should read: for every *odd* n. Dr. A. Brauer in Berlin has informed us that he has been able to show that theorem 16 holds for all $n > 6$.

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